ARCH and GARCH models

(Updated Spring 2021)

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Empirical Financial Econometrics

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ARCH & GARCH models

- ARCH & GARCH introduction
- An explicit model for squared error
- ARCH(q) Model
- Forecasting with ARCH models
- Generalized ARCH (GARCH)

ARCH & GARCH

- Stylized facts for stocks & exchang rate returns
  1. The volatility of returns is not constant.
  2. There is volatility persistence: high volatility periods and low volatility periods $\Longrightarrow$ led to development of ARCH & GARCH.

- ARCH = Autoregress Cconditional Heteroskedasticity
  GARCH = Generalized ARCH

Terminology

- Unconditional Homoskedasticity = Constant variance. e.g. $Var(\varepsilon) = \sigma^2$ for all t.

- Unconditional Heterokedasticity = Non-constant variance. e.g.

$$Var(\varepsilon_t) = \begin{cases} \sigma_1^2, & t < s \\ \sigma_2^2, & t \geq s \end{cases}$$

.

- Conditional Homoskedasticity = Constant conditional variance. e.g. $Var_t(\varepsilon_{t+1}) = \sigma^2$ for all t.

- Conditional Heterokedasticity = Non - constant conditional variance. e.g.

$$Var_t(\varepsilon_{t+1}) = \begin{cases} \sigma_1^2, & \varepsilon_t > 0 \\ \sigma_2^2, & \varepsilon_t \leq 0 \end{cases} .$$

Solving for the conditional variance ($var_{t-1}(y_t)$ for an ARMA(p,q))

- It is easier than you think

$$y_t = \underbrace{\alpha_0 + \sum_{i=1}^{p} \alpha_i y_{t-i} + \sum_{i=1}^{q} \beta_i \varepsilon_{t-i}}_{C} + \underbrace{\varepsilon_t}_{R}$$

$$\Longrightarrow \boxed{Var_{t-1}(y_t) = Var_{t-1}(\varepsilon_t) = E_{t-1}\varepsilon_t^2 - \underbrace{(E_{t-1}\varepsilon_t)^2}_{0} = E_{t-1}\varepsilon_t^2}.$$

- Conclusion: To model conditional variance of y_t need only model conditional variance of ε_t and this in turn means modeling $E_{t-1}\varepsilon_t^2$, i.e. modeling ε_t^2.

The ARCH concept

- We want to model the conditional mean of ε_t^2:

$$E_{t-1}\left[\varepsilon_t^2\right] = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 \quad \text{(An autoregressive model for } \varepsilon_t^2\text{)} \tag{1}$$

- Intuition: suppose ε_t is the (innovation of the) return on the TSE index. When there is a large movement today (ε_t^2 large), this is usually followed by large movement the next day (ε_{t+1}^2 large).

  e.g. If stocks crashed yesterday, ($\varepsilon_{t-1} << 0$), then today is unlikely to be a calm day.

How do we turn this into an explicit model for ε_t^2 ?

- First Inclination: Additive error

$$\varepsilon_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + v_t, \ \ v_t \sim WN(0, \sigma_v^2) \tag{2}$$

.

This satisfies (1) since it implies

$$E_{t-1}\varepsilon_t^2 = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2$$

.

But: what if v_t is large negative?

$\Longrightarrow$ could imply $\varepsilon_t^2 < 0$
$\Longrightarrow$ And, that's just not possible

An explicit model for ε_t^2: second try

- Second Try: Multiplicative Error

$$\varepsilon_t^2 = v_t^2(\alpha_0 + \alpha_1 \varepsilon_{t-1}^2) \tag{3}$$

$$E_{t-1} v_t = 0, \ \boxed{E_{t-1} v_t^2 = 1} \tag{4}$$

- Now if $\alpha_0 > 0$ and $\alpha_1 \geq 0$, this ensures $\varepsilon_t^2 > 0$.
- But, does ε_t^2 satisfy (1) ? Let's check!

$$\varepsilon_t^2 = v_t^2(\alpha_0 + \alpha_1 \varepsilon_{t-1}^2)$$

$$\begin{aligned} E_{t-1}\varepsilon_t^2 &= E_{t-1}\left[v_t^2(\alpha_0 + \alpha_1 \varepsilon_{t-1}^2)\right] \\ &= \underbrace{E_{t-1}[v_t^2]}_{=1 \text{ by assumption}} (\alpha_0 + \alpha_1 \varepsilon_{t-1}^2) \\ &= \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 \end{aligned}$$

- So, yes, (3) does satisfy (1)

An explicit model for ε_t^2: second try Cont.

- Although we may be more comfortable with additive error terms, the multiplicative errors work better in this context.
- Note also that the assumptions of (4): $E_{t-1}v_t = 0$ and $E_{t-1}v_t^2 = 1$ imply that $v_t \sim WN(0,1)$.
- Proof:
  1. already shown in past lectures that $E_{t-1}v_t = 0 \Longrightarrow E[v_t] = 0$ & $cov(v_t, v_{t+j}) = 0$ for $j \neq 0$.
  2. To show that $E[v_t^2] = 1$, we have

$$E[v_t^2] = E\underbrace{E_{t-1}v_t^2}_{=1} = E[1] = 1.$$

- So, yes (3) does satisfy (1). Equation (3) is an ARCH(1) model.

ARCH(q) Model

- ARCH(q) Model:

$$\varepsilon_t^2 = v_t^2(\alpha_0 + \textstyle\sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2) \quad (5a)$$

$$E_{t-1} v_t = 0 \text{ and } E_{t-1} v_t^2 = 1 \quad (5b)$$

$$\alpha_i \geq 0, \; i = 0, 1, 2, ..., q. \quad (5c)$$

Note that that the conditional heteroskedasticity is essentially modelled by an autoregression in ε_t^2.

- Coefficient restrictions:
Note that $\varepsilon_t^2 \geq 0$. So the right hand side (RHS) of (5) cannot ever imply $\varepsilon_t^2 < 0$.
This mandates (5c).

How is ARCH applied?

1. Can be applied directly, i.e.

$$y_t = \varepsilon_t \qquad \varepsilon_t = v_t\sqrt{\alpha_0 + \sum_{i=1}^{q}\alpha_i\varepsilon_{t-i}^2}.$$

   An e.g. might be an exchange rate return.

2. It can be applied to a series with a mean

$$y_t = \mu + \varepsilon_t; \qquad \varepsilon_t = v_t\sqrt{\alpha_0 + \sum_{i=1}^{q}\alpha_i\varepsilon_{t-i}^2}.$$

   An e.g. might be a stock return.

3. It can be applied to describe the residual of a regression on ARMA model

$$e.g.\ y_t = \alpha_0 + \alpha_1 y_{t-1} + \varepsilon_t$$
$$\varepsilon_t = v_t\sqrt{\alpha_0 + \sum_{i=1}^{q}\alpha_i\varepsilon_{t-i}^2}$$
$$E_{t-1}v_t = 0;\ E_{t-1}v_t^2 = 1.$$

How is ARCH applied? Cont.

- So we can model & forecast both the conditional mean & the conditional variance of the process, y_t.
- Recall that:

$$Var_{t-1}(y_t) = E_{t-1}[\varepsilon_t^2] = \alpha_0 + \sum_{i=1}^{q} \varepsilon_{t-i}^2$$

.

- When we considered the AR(1) before, we assumed $\varepsilon_t \sim WN(0, \sigma^2)$,
- Now we assume $\varepsilon_t \sim ARCH(q)$

Is ARCH compatible with ARMA and WN Errors?

- This begs the question: Is $\varepsilon_t \sim ARCH(q)$ still also white noise?
- To keep the algebra simple, let's work with $q = 1$,

$$\varepsilon_t = v_t \sqrt{(\alpha_0 + \alpha_1 \varepsilon_{t-1}^2)}$$

$$E_{t-1} v_t = 0; \; E_{t-1} v_t^2 = 1.$$

1. We first show $E_{t-1}\varepsilon_t = 0$:

$$E_{t-1}\varepsilon_t = E_{t-1}\{\underset{R}{v_t} \underbrace{\sqrt{\alpha_0 + \alpha_1 \varepsilon_{t-1}^2}_{NR}}\}$$

$$= \underbrace{E_{t-1}[v_t]}_{=0} \sqrt{\alpha_0 + \alpha_1 \varepsilon_{t-1}^2}$$

$$= 0.$$

2. Recall that $E_{t-1}\varepsilon_t = 0$ implies both:
(i) $E[\varepsilon_t] = 0$ and
(ii) $cov(\varepsilon_t, \varepsilon_{t+j}) = 0$ for $j \neq 0$.

Is ARCH compatible with ARMA and WN Errors? cont.

- Next, calculate $Var(\varepsilon_t) = E[\varepsilon_t^2]$:

$$
\begin{aligned}
E[\varepsilon_t^2] &= E\left[E_{t-1}\varepsilon_t^2\right] \qquad \text{Using L.I.E.} \\
&= E\left[\alpha_0 + \alpha_1 \varepsilon_{t-1}^2\right] \\
&= \alpha_0 + \alpha_1 E[\varepsilon_{t-1}^2].
\end{aligned}
$$

So,

$$E[\varepsilon_t^2] = \alpha_0 + \alpha_1 E[\varepsilon_{t-1}^2]$$

Is ARCH compatible with ARMA and WN Errors? cont.

- Now use the lag operator to solve for $E[\varepsilon_t^2]$:[1]

$$\begin{aligned} E[\varepsilon_t^2] &= \alpha_0 + \alpha_1 L E[\varepsilon_t^2] \\ E[\varepsilon_t^2] - \alpha_1 L E[\varepsilon_t^2] &= \alpha_0 \\ (1 - \alpha_1 L) E[\varepsilon_t^2] &= \alpha_0 \\ E[\varepsilon_t^2] &= \frac{(1 - \alpha_1 L)}{\alpha_0} \\ &= \frac{(1 - \alpha_1)}{\alpha_0} \end{aligned}$$

- For the last step recall that the lag operator has no effect on the constant

[1] If we knew ε_t was stationary, we could set $E[\varepsilon_t^2] = E[\varepsilon_{t-1}^2]$ and solve for $E[\varepsilon_t^2]$. But, we haven't established yet whether ε_t is stationary.

Is ARCH compatible with ARMA and WN Errors? cont.

- So, for $\alpha_1 < 1$, we have established that $\varepsilon_t \sim WN(0, \frac{\alpha_0}{1-\alpha_1})$.
- Note that this implies that ε_t is also covariance stationary.
- And that y_t will be covariance stationary under the usual condition (e.g. $|\alpha_1| < 1$).
- So we model the conditional variance ARCH and still model the conditional mean in the same way as before.
- Robert Engle introduced ARCH in a 1982 publication & was awarded a Nobel prize in 2003.

Forecasting with ARCH models

- Consider buying stock at time t and selling at time $t+1$.
- If you are risk averse, you may care about the variance of the stock return, say y_{t+1}.
- But why use unconditional variance?
  That would throw away all info you have about recent events.
- Use conditional variance instead: $Var_t(y_{t+1}) = E_t[\varepsilon_{t+1}^2]$.
- If $\varepsilon_t \sim ARCH(q)$, then $Var_t(y_{t+1}) = E_t[\varepsilon_{t+1}^2] = \alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t+1-i}^2$.
- So estimate or forecast of $Var_t(y_{t+1})$ is
  $\widehat{Var_t(y_{t+1})} = \widehat{\alpha}_0 + \sum_{i=1}^{q} \widehat{\alpha}_i \widehat{\varepsilon}_{t+1-i}^2$.

How well did we predict $Var_t(y_{t+1})$?

- How well did we predict $Var_t(y_{t+1})$?
- The problem in answering this is that we don't actually observe $Var_t(y_{t+1})$. In econometric terminology, it's latent.
- So we can't compare actual & predicted values to evaluate the forecast.
- On the other hand, ARCH also provides a prediction for ε^2_{t+1} and $\widehat{\varepsilon}^2_{t+1}$ is observable. So we can compare observed values of $\widehat{\varepsilon}^2_{t+1}$ to $E_{t-1}\varepsilon^2_t$, i.e.[2]

$$E_{t-1}\varepsilon^2_{t+1} = \alpha_0 + \sum_{i=1}^{q}\alpha_i\varepsilon^2_{t-i}$$
$$\widehat{\varepsilon^2}_{t+1|t} = \widehat{\alpha}_0 + \sum_{i=1}^{q}\alpha_i\widehat{\varepsilon}^2_{t-1}.$$

- In your forecast projects, you may consider using ARCH or GARCH to forecast squared stock or exchange rate returns.

[2] Here $\widehat{\varepsilon^2}_{t+1|t}$ denotes the time-t ARCH forecast of ε^2_{t+1}, whereas $\hat{\varepsilon}^2_t$ refers to the fitted residual at time-t.

Generalized ARCH (GARCH)

$$\varepsilon_t = v_t\sqrt{h_{t|t-1}} \tag{6a}$$

$$h_{t|t-1} = \alpha_0 + \sum_{i=1}^{q}\alpha_i\varepsilon_{t-i}^2 + \sum_{i=1}^{p}\beta_i h_{t-i|t-i-1} \tag{6b}$$

$$E_{t-1}v_t = 0 \tag{6c}$$

$$E_{t-1}v_t^2 = 1 \tag{6d}$$

- Equations (6a)-(6d) model ε_t as a GARCH(p,q).
- From (6b) can see that $h_{t|t-1}$ is known at time $t-1$.
- $E_{t-1}\varepsilon_t = \underbrace{(E_{t-1}v_t)}_{0}\underbrace{\sqrt{h_{t|t-1}}}_{C} = 0$
- $E_{t-1}\varepsilon_t^2 = \underbrace{(E_{t-1}v_t^2)}_{1}h_{t|t-1} = h_{t|t-1}$

Generalized ARCH (GARCH) Cont.

- $Var_{t-1}\varepsilon_t^2 = \underbrace{(E_{t-1}\varepsilon_t^2)}_{h_{t|t-1}} - \underbrace{(E_{t-1}\varepsilon_t)^2}_{0} = h_{t|t-1}$
- $h_{t|t-1}$ is the conditional variance of ε_t and (6b) models it in the style of an ARMA(p,q) model with AR components $\beta_i h_{t-i|t-i-1}$ and MA components $\alpha_i \varepsilon_{t-i}^2$.
- $h_{t|t-1}$ often just called "h_t" because it is conditional variance of ε_t. However, it is realized at time $t-1$, because it is time $t-1$ conditional variance. I use $h_{t|t-1}$ to remind us of this.
- Can write it as generalization of ARCH by combining (6a) and (6b) to get:

$$\varepsilon_t^2 = v_t^2 h_{t|t-1} \tag{7}$$

$$\Longrightarrow \boxed{\varepsilon_t^2 = v_t^2\left(\alpha_0 + \sum_{i=1}^{q} \alpha_i \varepsilon_{t-i}^2 + \sum_{i=1}^{p} \beta_i h_{t-i|t-i-1}\right)}$$

GARCH and ARCH relation

- If $\beta_i = 0, \ i = 1, ..., p$, then (7) specializes to an ARCH(q)
  $\Longrightarrow GARCH(0, q) = ARCH(q)$.
- Advantage of GARCH over ARCH is similar to advantage of ARMA over AR:
  Parsimony: more flexible model with fewer parameters