

Probability

Eco 2740: Economic Statistics
Fall, 2019. (Chapter 6)

Approaches to Assigning Probabilities...

There are four ways to assign a probability, $P(O_i)$, to an outcome, O_i , namely:

Classical approach: based on equally likely events.

Relative frequency: based on experimentation or historical data.

Subjective Approach based on (subjective) judgment.

Bayesian approach: based on combination of subjective assessment with relative frequency

Classical Approach...

If an experiment has n equally likely outcomes, this method would assign a probability of $1/n$ to each outcome. It is necessary to determine the number of possible outcomes.

| | |
|----------------|--|
| Experiment: | Rolling a <i>die</i> |
| Outcomes | {1, 2, 3, 4, 5, 6} |
| Probabilities: | Each sample point has a $1/6$ chance of occurring. |

Classical Approach...

Experiment: Rolling two dice and observing the total

Outcomes: $\{2, 3, \dots, 12\}$

Examples:

$$P(2) = 1/36$$

$$P(6) = 5/36$$

$$P(10) = 3/36$$

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|----|----|----|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
| 2 | 3 | 4 | 5 | 6 | 7 | 8 |
| 3 | 4 | 5 | 6 | 7 | 8 | 9 |
| 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 |

Relative Frequency Approach...

Bits & Bytes Computer Shop tracks the number of desktop computer systems it sells over a month (30 days):

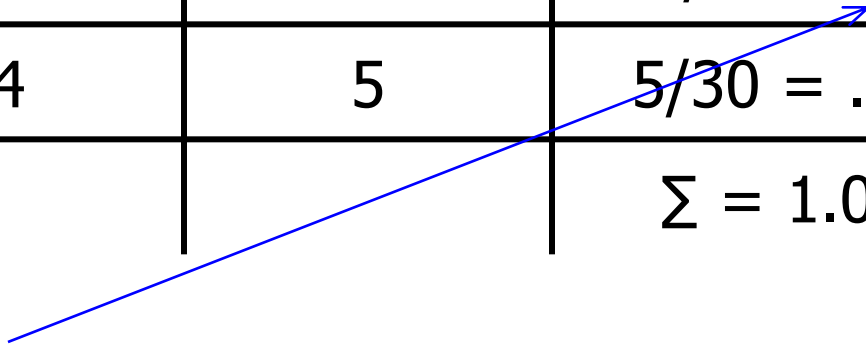
For example,
10 days out of 30
2 desktops were sold.

| Desktops Sold | # of Days |
|---------------|-----------|
| 0 | 1 |
| 1 | 2 |
| 2 | 10 |
| 3 | 12 |
| 4 | 5 |

From this we can construct
the probabilities of an event
(i.e. the # of desktop sold on a given day)...

Relative Frequency Approach...

| Desktops Sold | # of Days | Desktops Sold |
|---------------|-----------|-----------------|
| 0 | 1 | $1/30 = .03$ |
| 1 | 2 | $2/30 = .07$ |
| 2 | 10 | $10/30 = .33$ |
| 3 | 12 | $12/30 = .40$ |
| 4 | 5 | $5/30 = .17$ |
| | | $\Sigma = 1.00$ |



“There is a 40% chance Bits & Bytes will sell 3 desktops on any given day”

Subjective Approach...

“In the subjective approach we define probability as the degree of belief that we hold in the occurrence of an event”

- E.g. What is the probability that you and a prospective roommate will get along well?
- E.g. After a first date, what is the probability that it will turn into a long term relationship?
- It may be hard to place scientifically based objective probabilities on these events, but you may still be able to develop your own subjective probabilities to help guide your decisions.

Subjective Approach,...

- **E.g. Survivalism/Prepper Movement** – a movement that encourages the stockpiling of food and water, the building of underground structures, etc. to prepare for possible disasters, such as societal collapse, nuclear war, etc.
- Because major disasters of the type preppers have in mind are so infrequent it is hard to place an objective probability on them.
- Left to rely on subjective probabilities.
- Preppers likely put a higher subjective probability on the possibility of such disasters than most of us.

Events and their Probabilities

- An Event is something specific that might (or might not) happen. E.g. It rains tomorrow or I roll two sixes.
- Often we assign it a capital letter:
E.g. $A = \{\text{It Rains Tomorrow}\}$. $B = \{\text{Roll Two Sixes}\}$
- And a probability (P):
E.g. $P(A) = 0.25$ or $P(B) = 1/36$.
- Events can be simple (roll a three with one die) or complex (e.g a collection of simple events, such as two dice add to six)

Venn Diagrams

- Picture the rectangle bellow as the universe of all possible events that could conceivably occur (e.g. it contains all possible weather outcomes, or all possible throws of two dice).

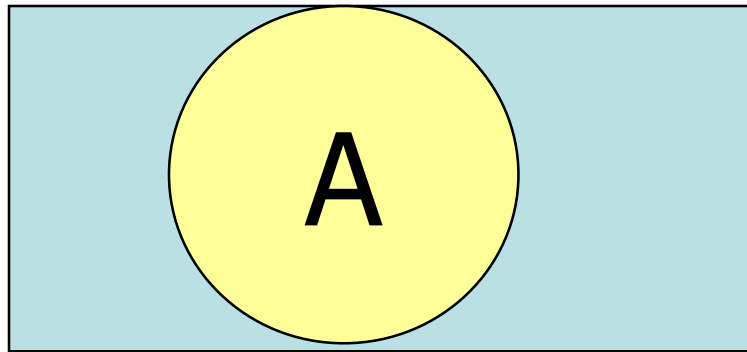


- Since one of these outcomes must happen our probability of “landing in the rectangle” is 1. We can visualize:

$$P(\text{Rectangle}) = \text{Area of Rectangle} = 1$$

Venn Diagrams, continued

- Event A is just one possible outcome. So it is contained inside the rectangle.



← Venn Diagram

- We can visualize:

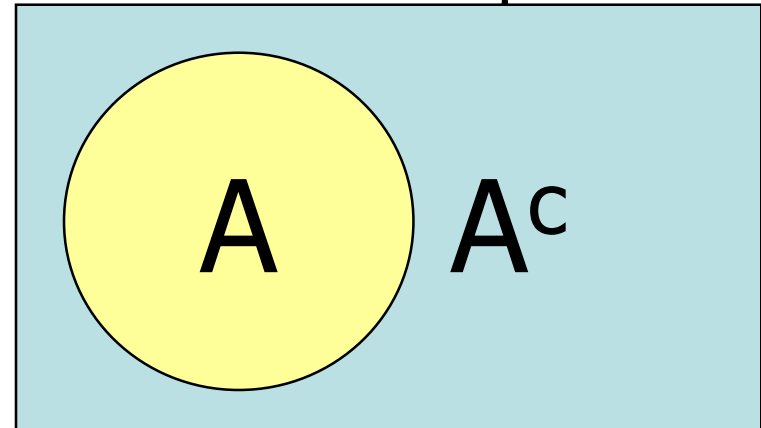
$$\begin{aligned} P(A) &= (\text{Area of } A) / (\text{Area of Rectangle}) = P(A) / P(\text{Rectangle}) \\ &= P(A) / 1 = P(A) \end{aligned}$$

Complement of an Event...

The *complement of event* A is defined to be the event consisting of all outcomes that are “not in A ”.

Complement of A is denoted by A^c

The Venn diagram below illustrates the concept of a complement.



$$P(A) + P(A^c) = 1$$

$$P(A^c) = 1 - P(A)$$

Complement of an Event...

E.g. the rectangle stores all the possible tosses of 2 dice. Let A = tosses totaling either 2, 6, or 10.

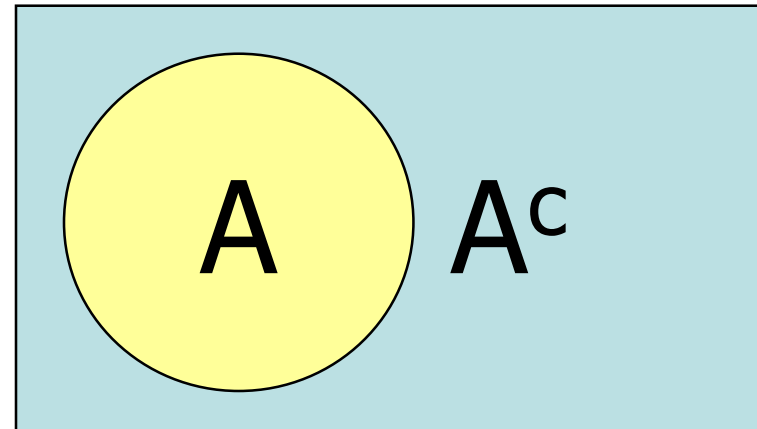
A^C = NOT A = tosses not totaling 2, 6, or 10

= tosses totaling 3,4,5,7,8,9,11, or 12

$$P(A) = (1+5+3)/36 = 9/36 = 1/4$$

$$P(A^C) = 1 - P(A) = 1 - 1/4 = 3/4$$

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|----|----|----|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
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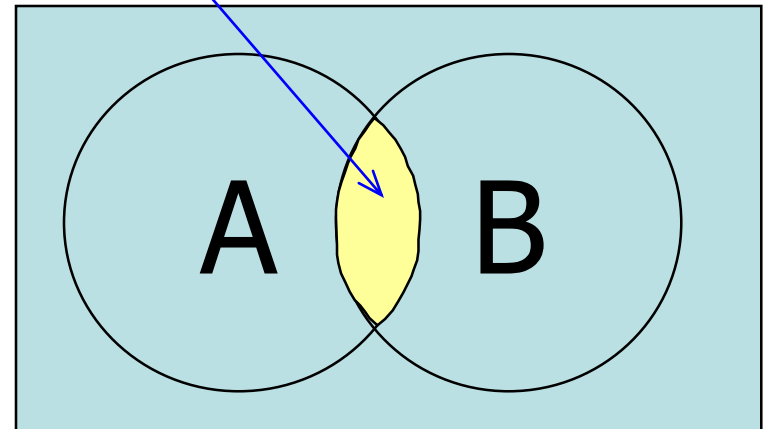
Intersection & Joint Probability of Two Events

The *intersection of events* A and B is the set of all sample points that are in both A and B.

The intersection is denoted: **A and B** (Alternatively: $A \cap B$)

The *joint probability* of A and B is the probability of the intersection of A and B, i.e. **P(A and B)**

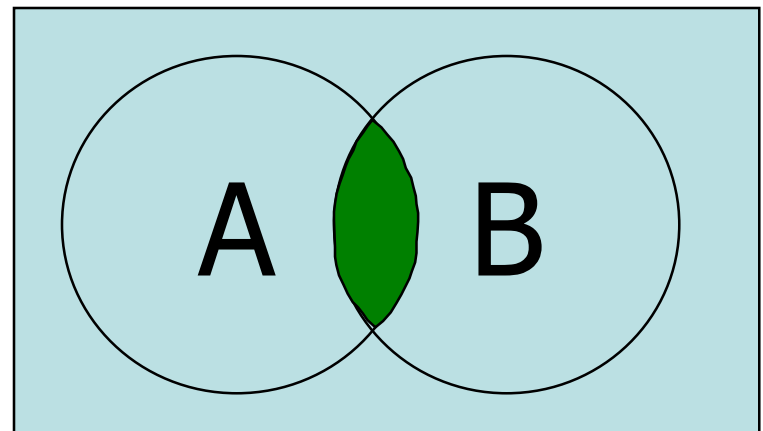
It is the probability that both A and B happen



Intersection of Two Events...

- E.g. $A = \text{first toss is } 1 = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)\}$
- $B = \text{second toss is } 5 = \{(1,5), (2,5), (3,5), (4,5), (5,5), (6,5)\}$
- The intersection is $A \text{ and } B = \{(1,5)\}$
- The *joint probability* of A and B = $P(A \text{ and } B) = 1/36$

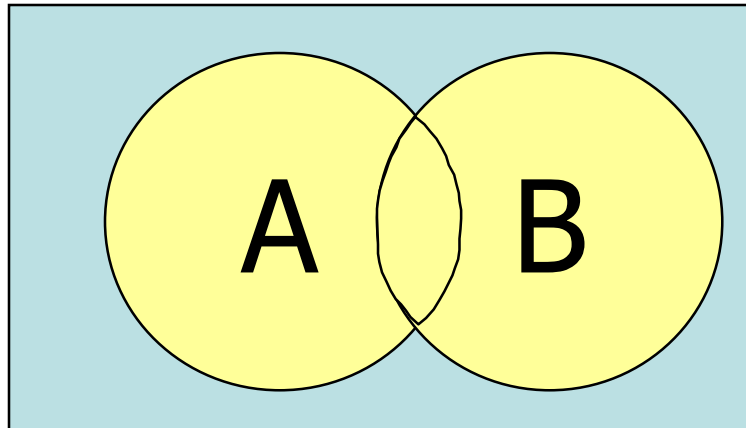
| | 1 | 2 | 3 | 4 | 5 | 6 |
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| 5 | 6 | 7 | 8 | 9 | 10 | 11 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 |



Union of Two Events...

The *union of two events* A and B, is the event containing all sample points that are in A or B or both:

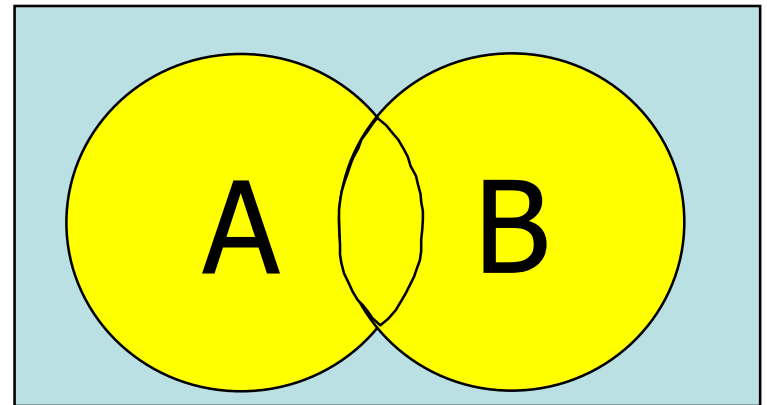
Union of A and B is denoted: **A or B** (Alternatively: $A \cup B$)



Union of Two Events...

- E.g. $A = \text{first toss is } 1 = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)\}$
- $B = \text{second toss is } 5 = \{(1,5), (2,5), (3,5), (4,5), (5,5), (6,5)\}$
- Union of A and B = A **or** B = everything in A and everything in B
 $= \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (2,5), (3,5), (4,5), (5,5), (6,5)\}$
- $P(A \text{ or } B) = 11/36$

| | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|----|----|----|
| 1 | 2 | 3 | 4 | 5 | 6 | 7 |
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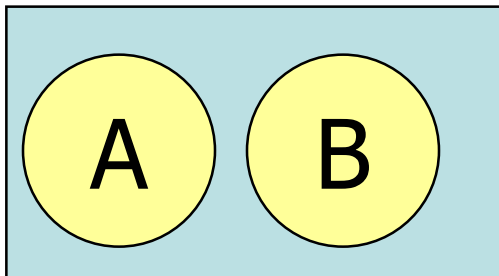
Mutually Exclusive Events...

When two events are *mutually exclusive* (that is the two events cannot occur together), their joint probability is 0, hence:

Mutually exclusive; no points in common...

For example A = tosses totaling 7 and B = tosses totaling 11

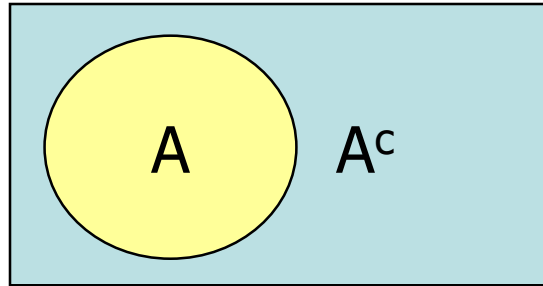
Joint Probability = $P(A \text{ and } B) = P(\text{Empty Set}) = 0$



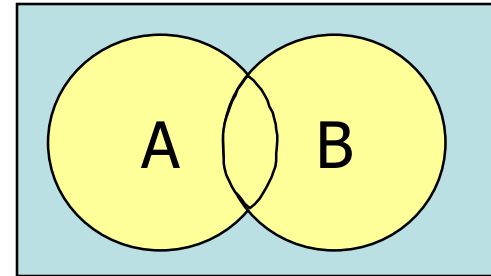
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Basic Relationships of Probability...

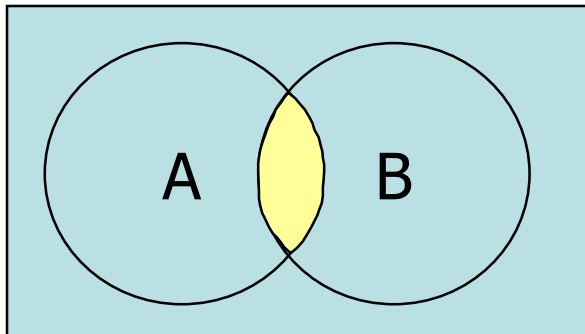
Complement of Event



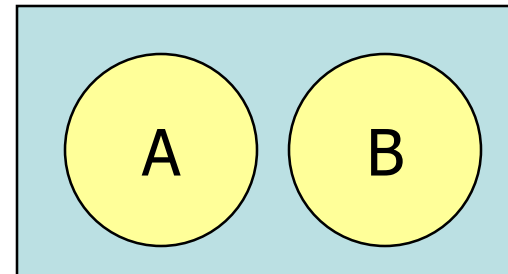
Union of Events



Intersection of Events



Mutually Exclusive Events



Example 6.1...

Why are some mutual fund managers more successful than others? One possible factor is where the manager earned his or her MBA. The following table compares mutual fund performance against the ranking of the school where the fund manager earned their MBA:

| | Mutual fund outperforms the market | Mutual fund doesn't outperform the market |
|------------------------|------------------------------------|---|
| Top 20 MBA program | .11 | .29 |
| Not top 20 MBA program | .06 | .54 |

E.g. This is the probability that a mutual fund outperforms **AND** the manager was in a top-20 MBA program; it's a **joint probability**.

Example 6.1...

Alternatively, we could introduce shorthand notation to represent the events:

A_1 = Fund manager graduated from a top-20 MBA program

A_2 = Fund manager did not graduate from a top-20 MBA program

B_1 = Fund outperforms the market

B_2 = Fund does not outperform the market

| MBA \ Perform | B_1 | B_2 |
|---------------|------------|------------|
| A_1 | .11 | .29 |
| A_2 | .06 | .54 |

E.g. $P(A_2 \text{ and } B_1) = .06$

= the probability a fund outperforms the market
and the manager isn't from a top-20 school.

Marginal Probabilities...

Marginal probabilities are computed by adding across rows and down columns; that is they are calculated in the *margins* of the table:

$$P(A_2) = .06 + .54$$

“what’s the probability a fund manager isn’t from a top school?”

| MBA \ Perform | B ₁ | B ₂ | P(A _i) |
|--------------------|----------------|----------------|--------------------|
| A ₁ | .11 | .29 | .40 |
| A ₂ | .06 | .54 | .60 |
| P(B _j) | .17 | .83 | 1.00 |

$$P(B_1) = .11 + .06$$

“what’s the probability a fund outperforms the market?”

BOTH margins must add to 1
(useful error check)

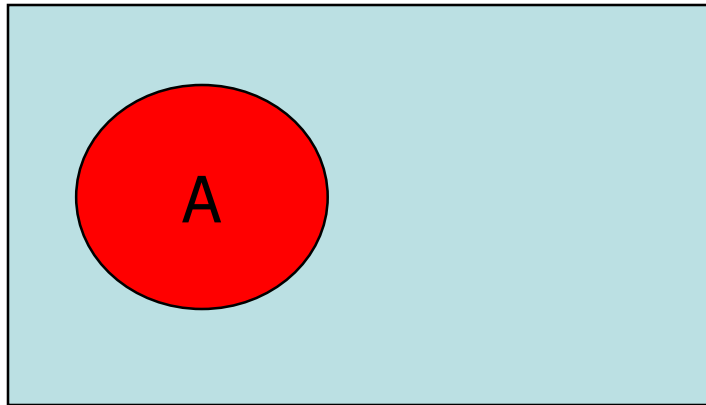
Conditional Probability...

Conditional probability is used to determine how two events are related; that is, we can determine the probability of one event *given* the occurrence of another related event.

Conditional probabilities are written as $P(A | B)$ and read as “the probability of *A given B*” and is calculated as:

$$P(A | B) = \frac{P(A \text{ and } B)}{P(B)}$$

Conditional Probability Formula - Intuition



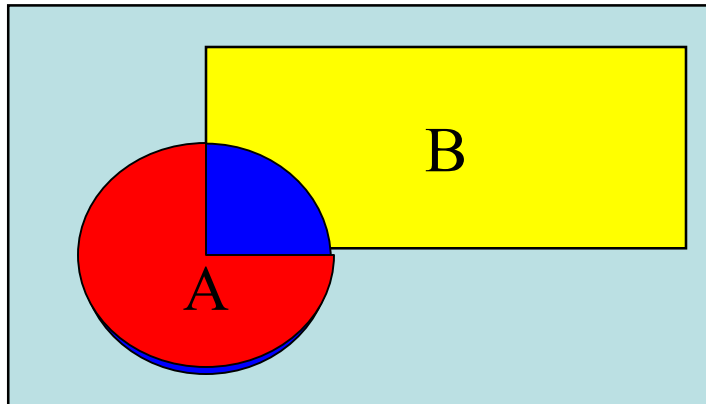
W/out knowing if B occurred:

$$P(A) = \frac{P(A)}{P(\text{Big Rectangle})}$$

$$= P(A)/1 = P(A)$$

Big Rectangle: all possible events before knowing B:

B = small rectangle = all possible events after knowing B occurred



$$P(A|B) = \frac{P(\text{part of A in B})}{P(\text{Small rectangle})}$$

$$= \frac{P(A \text{ and } B)}{P(B)}$$

Conditional Probability...

Again, the probability of an event *given* that another event has occurred is called a conditional probability...

$$P(A | B) = \frac{P(A \text{ and } B)}{P(B)}$$

$$P(B | A) = \frac{P(A \text{ and } B)}{P(A)}$$

Note how “A given B” and “B given A” are related...

Example 6.2 What's the probability that a fund will outperform the market (B_1) *given* that the manager graduated from a top-20 MBA program (A_1)?

| MBA \ Perform | B_1 | B_2 | $P(A_i)$ |
|---------------|------------|------------|-------------|
| A_1 | .11 | .29 | .40 |
| A_2 | .06 | .54 | .60 |
| $P(B_j)$ | .17 | .83 | 1.00 |

$$P(B_1 | A_1) = \frac{P(A_1 \text{ and } B_1)}{P(A_1)} = \frac{.11}{.40} = .275$$

Thus, there is a 27.5% chance that that a fund will outperform the market given that the manager graduated from a top-20 MBA program.

Distinguishing Joint & Conditional Probabilities

- **Joint Probability**: $P(\text{Top MBA and Outperform})$
 - Both top-MBA and outperform are unknown and random. What is the probability of both.
 - Pick a fund manager at random from pool of all fund managers: what is the probability that the manager you picked has both top-MBA and outperformance.
 - Out of all fund managers, what proportion have both top-MBA and outperformance
- **Conditional Probability**: $P(\text{Outperform} \mid \text{Top MBA})$:
 - Top MBA is known. Only outperform is random. What is probability of outperform given we know manager has top MBA.
 - Pick a fund manager at random from pool of all fund managers with top MBAs. What is probability he outperforms
 - Out of all fund managers with top MBA's what proportion outperform.

Independence...

One of the objectives of calculating conditional probability is to determine whether two events are related.

In particular, we would like to know whether they are *independent*, that is, if the probability of one event is *not affected* by the occurrence of the other event.

Two events A and B are said to be *independent* if

$$P(A|B) = P(A)$$

or

$$P(B|A) = P(B)$$

Independence...

For example, we saw that

$$P(B_1 | A_1) = .275$$

Recall that the marginal probability for B_1 was: $P(B_1) = 0.17$

Since $P(B_1|A_1) \neq P(B_1)$, B_1 and A_1 are **not independent** events.

Stated another way, they are ***dependent***. That is, the probability of one event (B_1) ***is affected*** by the occurrence of the other event (A_1).

Probability Rules and Trees...

We introduce three rules that enable us to calculate the probability of more complex events from the probability of simpler events...

The Complement Rule,

The Multiplication Rule, and

The Addition Rule

Complement Rule...

The *complement rule* gives us the probability of an event NOT occurring. That is:

$$P(A^C) = 1 - P(A)$$

For example, in the simple roll of a die, the probability of the number “1” being rolled is $1/6$. The probability that some number other than “1” will be rolled is $1 - 1/6 = 5/6$.

Example: Probability of Union w/out complement rule

Determine the probability that a fund outperforms (B_1)
or the manager graduated from a top-20 MBA program (A_1).

| | | Perform | | |
|-------|----------|------------|------------|-------------|
| | | B_1 | B_2 | $P(A_i)$ |
| MBA | | | | |
| A_1 | A_1 | .11 | .29 | .40 |
| | A_2 | .06 | .54 | .60 |
| | $P(B_j)$ | .17 | .83 | 1.00 |

$$P(A_1 \text{ or } B_1) = .11 + .06 + .29 = .46$$

Alternatively, using the complement rule

Take 100% and subtract off “when doesn't A₁ or B₁ occur”?

i.e. at A₂ and B₂

| | | | | |
|----------------|--------------------|----------------|----------------|--------------------|
| | | B ₁ | | |
| | Perform | | | |
| | MBA | B ₁ | B ₂ | P(A _i) |
| A ₁ | A ₁ | .11 | .29 | .40 |
| | A ₂ | .06 | .54 | .60 |
| | P(B _j) | .17 | .83 | 1.00 |

$$P(A_1 \text{ or } B_1) = 1 - P(A_2 \text{ and } B_2) = 1 - .54 = \mathbf{.46}$$

Multiplication Rule...

The *multiplication rule* is used to calculate the *joint probability* of two events. It is based on the formula for conditional probability defined earlier:

$$P(A | B) = \frac{P(A \text{ and } B)}{P(B)}$$

If we multiply both sides of the equation by $P(B)$ we have:

$$P(A \text{ and } B) = P(A | B) \cdot P(B)$$

Likewise,

$$P(A \text{ and } B) = P(B | A) \cdot P(A)$$

Special Case: If A and B are **independent** then:

$$P(A \text{ and } B) = P(A) \cdot P(B)$$

Example 6.5: Sampling without replacement

A graduate statistics course has seven male and three female students. The professor wants to select two students at random to help her conduct a research project. What is the probability that the two students chosen are female?

Let $A = \{\text{first student female}\}$ $B = \{\text{second student female}\}$

$$P(A) = 3/10 = .30 \quad (3 \text{ females out of } 10 \text{ students})$$

$$P(B | A) = 2/9 = .22 \quad (2 \text{ females left, } 9 \text{ student left})$$

$$\mathbf{P(A \text{ and } B) = P(B | A) \cdot P(A) = (3/10)(2/9) = 6/90 = .067}$$

Key Assumption: Same student can't be selected twice:
(sampling without replacement)

Example 6.6: Sampling With Replacement

In each of the next two classes the professor will pick one student at random to answer questions during that class. What is the probability that the two students chosen are female?

Let $A = \{\text{first student female}\}$ $B = \{\text{second student female}\}$

$P(A) = 3/10 = .30$ (3 females out of 10 students)

$P(B | A) = 3/10 = .30$ (3 females out of 10, again)

$P(A \text{ and } B) = P(B | A) \cdot P(A) = (3/10) (3/10) = .090$

Key Assumption: Same student can be selected twice:
(sampling with replacement)

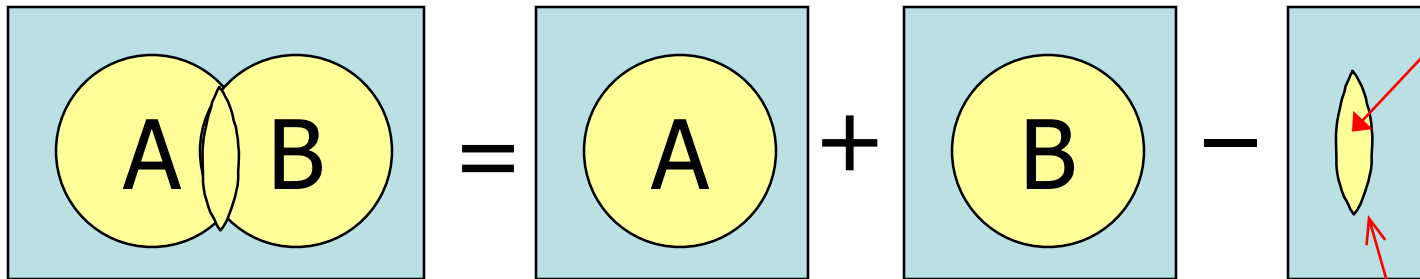
NOTE: A and B are independent in this problem

Addition Rule...

The *addition rule* is used to compute the probability of the union of A and B (A **or** B)

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

Subtract this term to avoid double counting



$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)$$

If A and B are mutually exclusive, then this term goes to zero

Example 6.7...

In a large city, two newspapers are published, the Sun and the Post. 22% of the city's households subscribe to the Sun and 35% subscribe to the Post. 6% of all households subscribe to both newspapers. What is the probability of selecting a household at random that subscribes to the Sun or the Post or both?

$$\begin{aligned} P(\text{Sun or Post}) &= P(\text{Sun}) + P(\text{Post}) - P(\text{Sun and Post}) \\ &= .22 + .35 - .06 = .51 \end{aligned}$$

“There is a 51% probability that a randomly selected household subscribes to one or the other or both papers”

Probability Trees

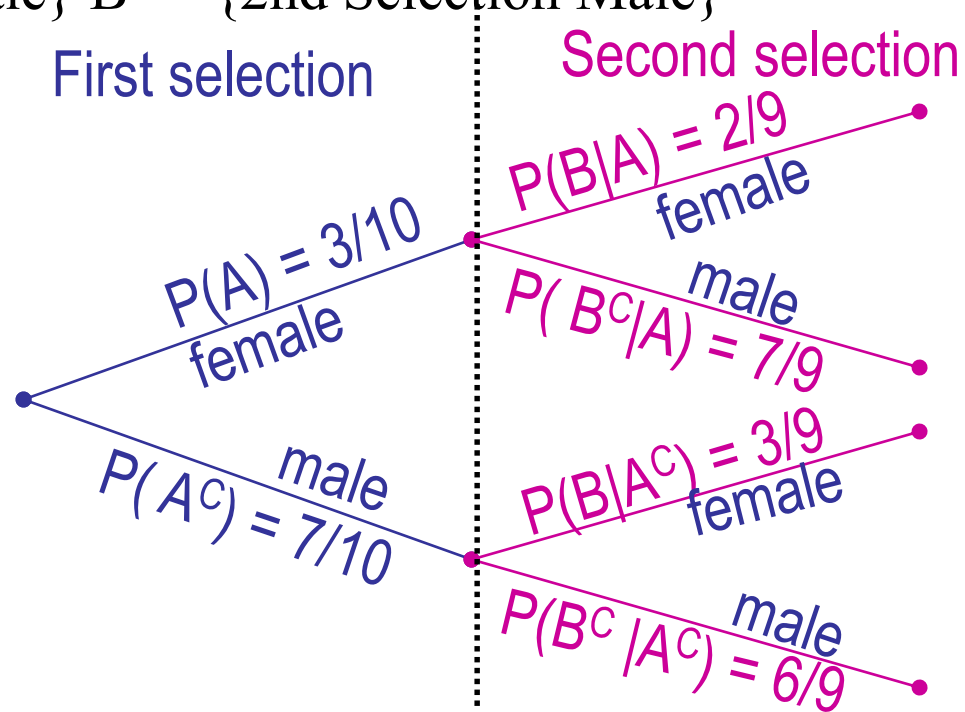
A visual method of applying the probability rules is the probability tree, wherein the events in an experiment are represented by lines. The resulting figure resembles a tree, hence the name. We will illustrate the probability tree with examples.

Example 6.5: Sampling without replacement

A graduate statistics course has seven male and three female students. The professor wants to select two students at random to help her conduct a research project. What is the probability that the two students chosen are female?

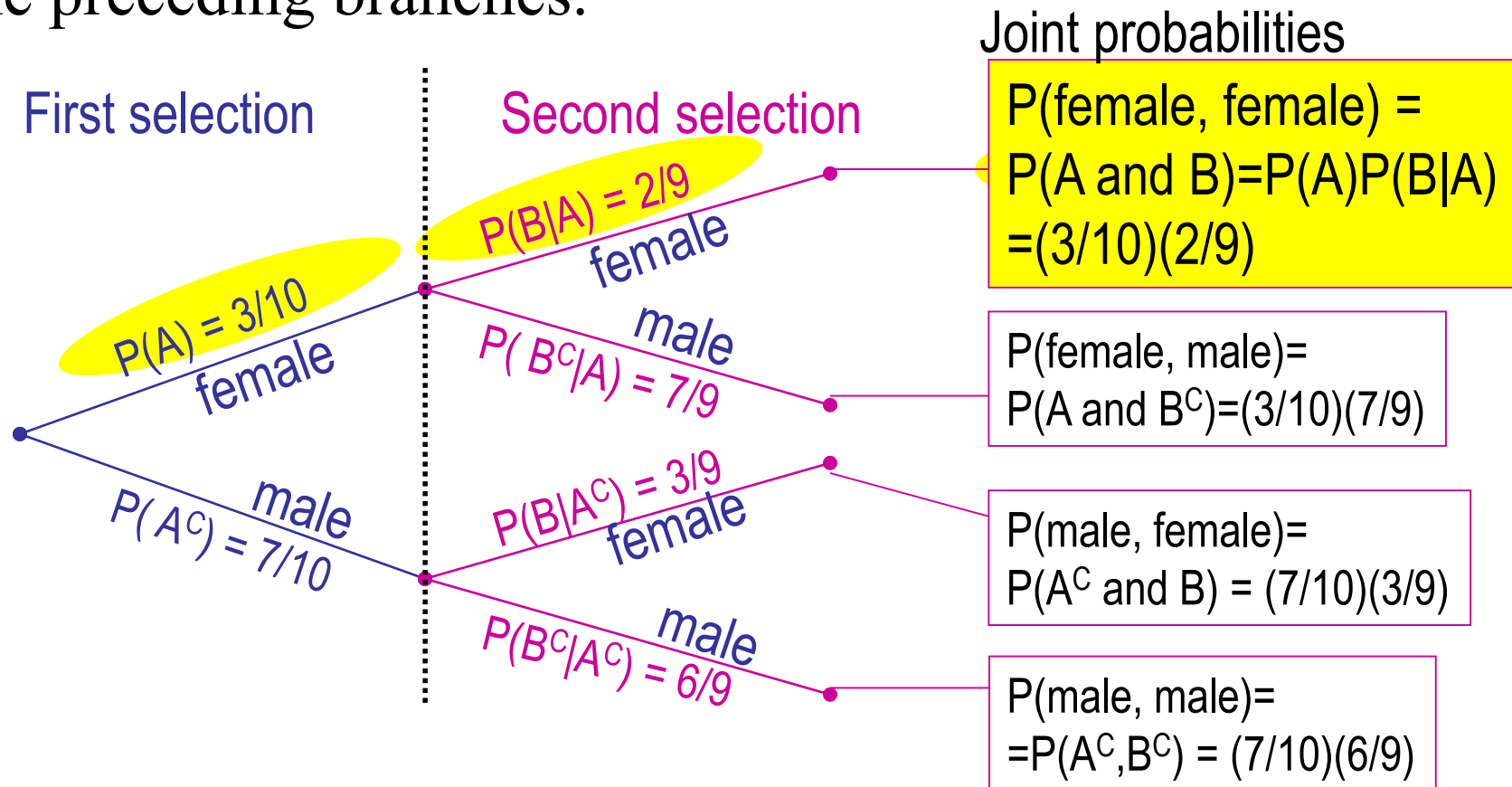
$A = \{\text{First Selection Female}\}$ $B = \{\text{2nd Selection Female}\}$

$A^C = \{\text{First Selection Male}\}$ $B^C = \{\text{2nd Selection Male}\}$



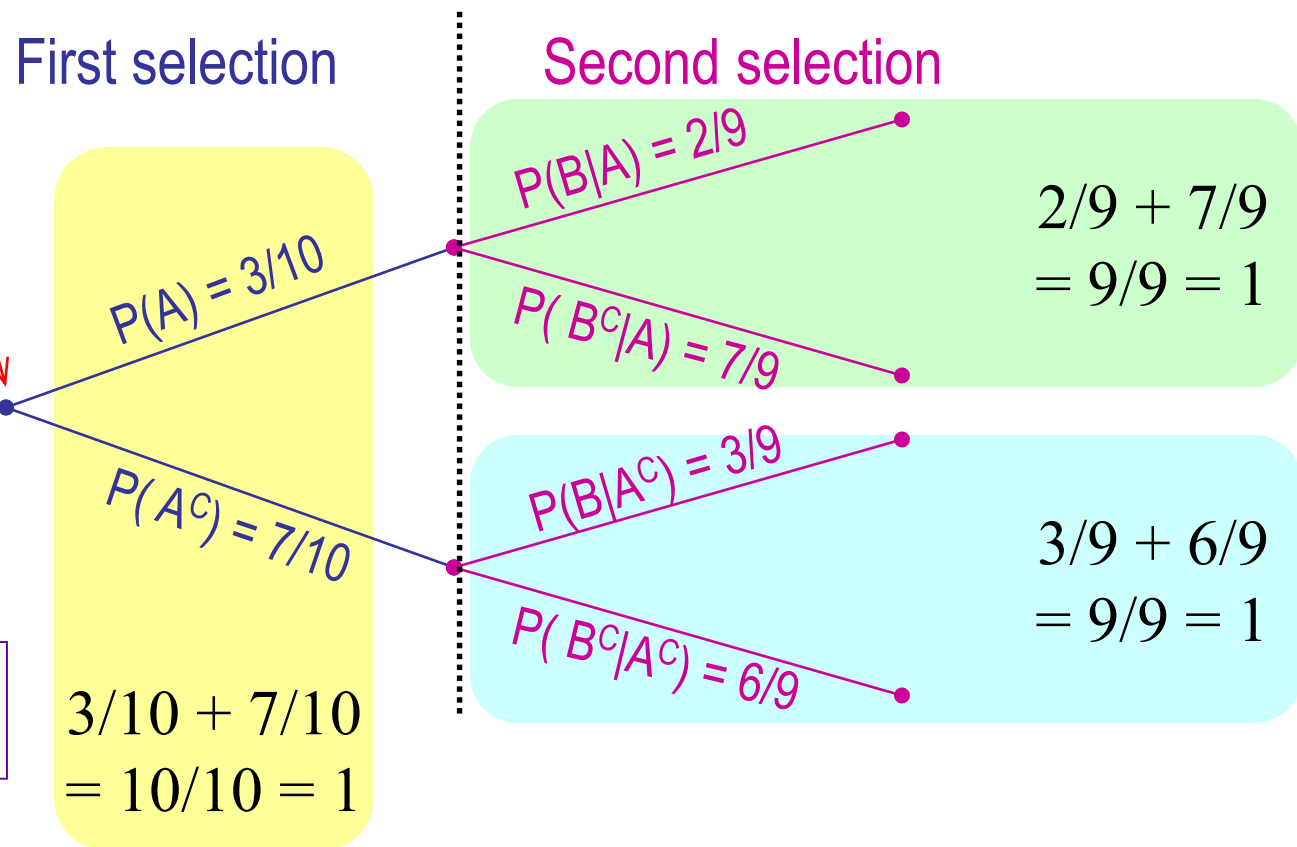
Probability Trees...

At the ends of the “branches”, we calculate *joint probabilities* as the **product** of the individual probabilities on the preceding branches.



Probability Trees...

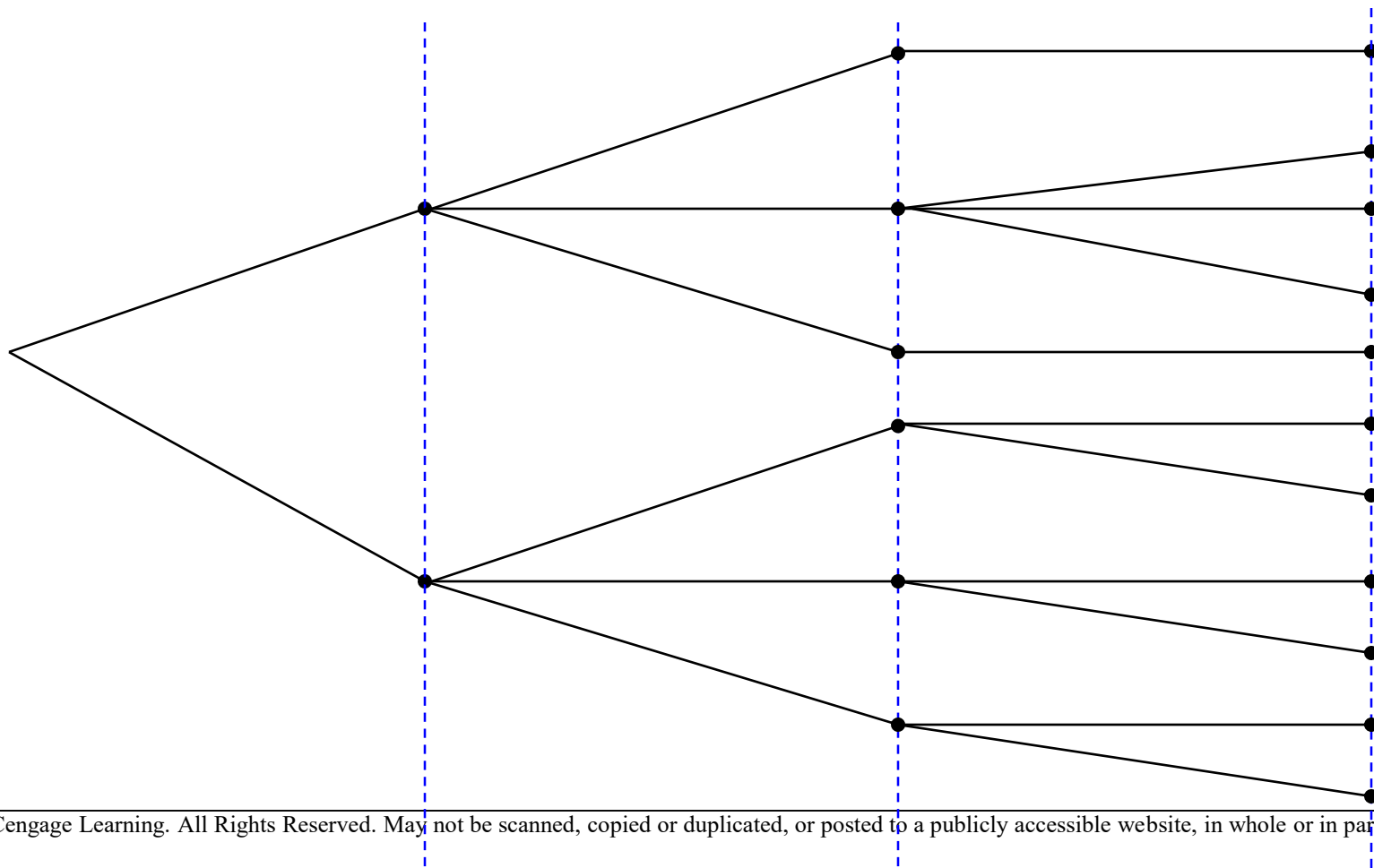
The probabilities associated with any set of branches from one “node” must add up to 1.00...



Handy way to check your work !

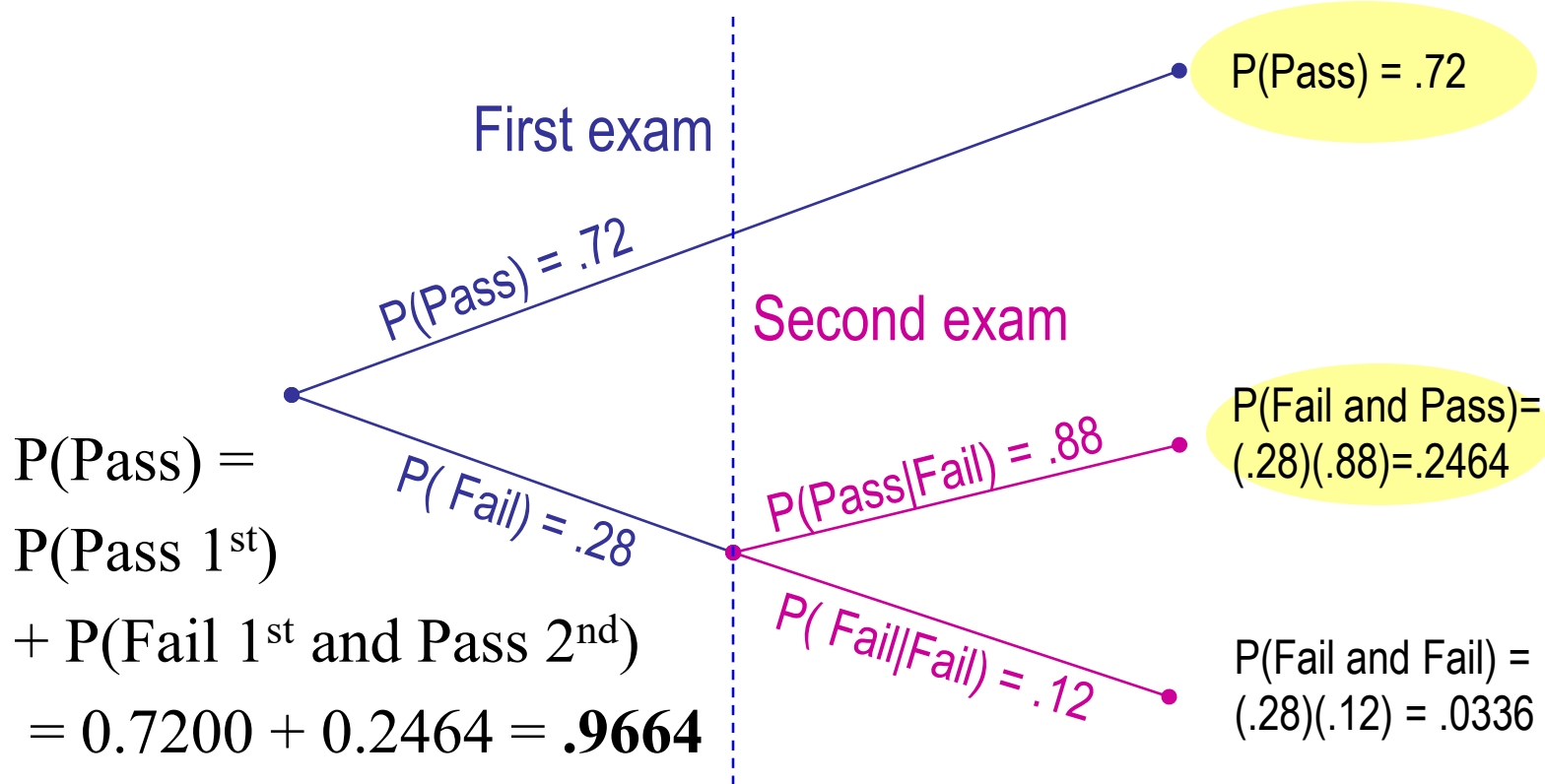
Probability Trees...

Note: there is no requirement that the branches splits be binary, nor that the tree only goes two levels deep, or that there be the same number of splits at each sub node...



Example 6.8

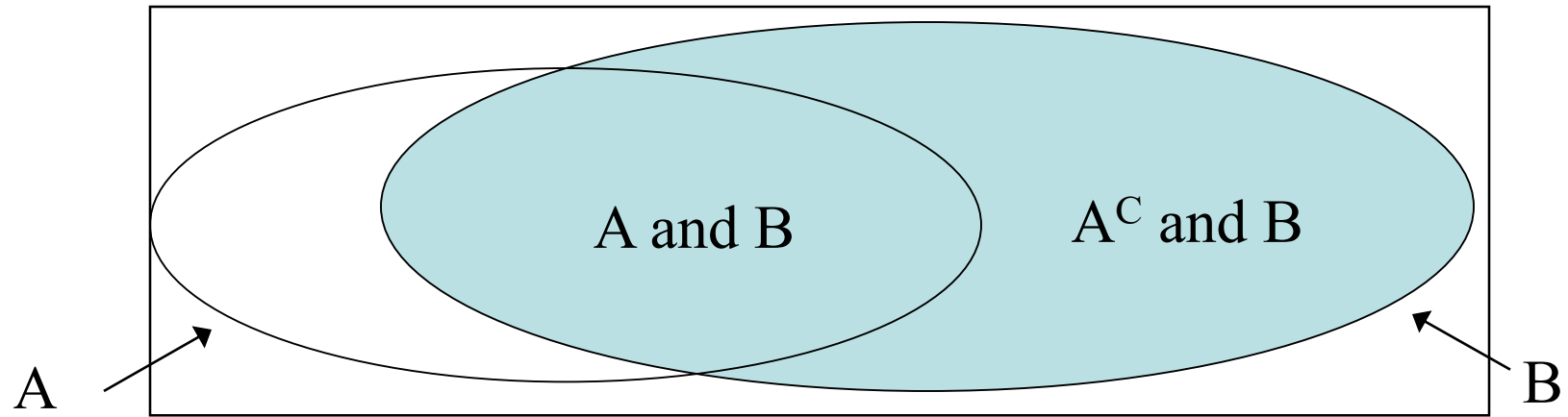
U.S. lawyers must pass a bar exam. Suppose pass rate for first-time test takers is 72%. They can re-write if they fail and 88% pass their second attempt. What is the probability that a randomly grad passes the bar?



The Law of Total Probability

The Simple Version (from Venn Diagram):

$$P(B) = P(A \text{ and } B) + P(A^C \text{ and } B)$$



The Complex Version

(combine simple version with multiplication rule):

$$P(B) = P(B|A)P(A) + P(B|A^C)P(A^C)$$

Example 6.9 – MBA prep course

Suppose that among GMAT scorers above 650, 52% took a prep course, whereas among scorers of less than 650 only 23% took a prep course. Overall, only 10% of applicants obtained a score of above 650. What is the probability that a randomly selected applicant has taken a preparatory class?

Let $A = \{\text{GMAT} \geq 650\}$, $B = \{\text{Took Preparatory Course}\}$

$$P(B | A) = 0.52 \quad P(B | A^C) = 0.23 \quad P(A) = 0.10$$

$$\begin{aligned} \text{Solve for: } P(B) &= P(A \text{ and } B) + P(A^C \text{ and } B) \\ &= P(B|A)P(A) + P(B|A^C)P(A^C) \end{aligned}$$

$$\begin{aligned} P(A^C) &= 1 - P(A) \\ &= 1 - 0.10 = 0.90 \end{aligned}$$

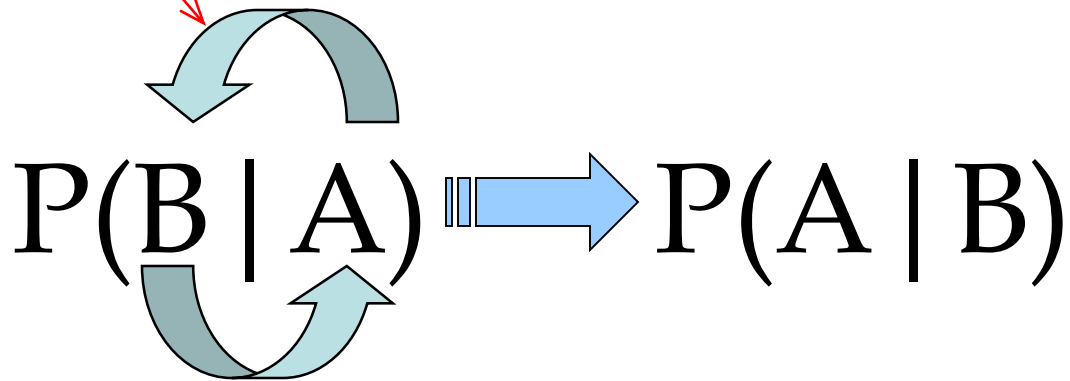
$$= (0.52)(0.10) + (0.23)(0.90) = 0.259$$

Bayes' Law...

Bayes' Law is named for Thomas Bayes, an eighteenth century mathematician.

In its most basic form, if we know $P(B | A)$,

we can apply Bayes' Law to determine $P(A | B)$



Bayes' Law Continued

- $$P(B|A) = \frac{P(A|B) P(B)}{P(A)} \qquad P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

- We can obtain $P(B|A)$ as follows:

$$\begin{aligned} P(B|A) &= \frac{(A \text{ and } B)}{P(A)} \quad [\text{apply definition of conditional probability}] \\ &= \frac{P(A | B) P(B)}{P(A)} \quad [P(A \text{ and } B) = P(A|B) P(B)] \end{aligned}$$

- Practice at home: obtain $P(A|B)$

Example 6.9 – Revisited

Suppose that among GMAT scorers above 650, 52% took a prep course, whereas among scorers of less than 650 only 23% took a prep course. Overall, only 10% of applicants obtained a score of above 650. What is the probability that an applicant that took the prep course scores above 650?

Let $A = \{\text{GMAT above 650}\}$, $B = \{\text{Preparatory Course}\}$

$$P(B | A) = 0.52 \quad P(B | A^c) = 0.23 \quad P(A) = 0.10$$

$$P(B) = 0.259 \quad (\text{already solved for it using law of total prob})$$

Now Solve for:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)} = \frac{(0.52)(0.10)}{0.259} = 0.201$$

Alternate Solution: Use the tree:

$$P(A | B) = \frac{P(A \text{ and } B)}{P(B)} = \frac{.052}{.259} = .201$$

