Extension to ARCH & GARCH model

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Empirical Financial Econometrics

Outlines

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Extension to ARCH & GARCH

- Integrated GARCH (IGARCH)
- ARCH-M model
- Threshold GARCH
- Exponential GARCH

Integrated GARCH (IGARCH) (Nelson 1990)

- Recall the Random Walk (RW) model

$$y_t = y_{t-1} + \varepsilon_t,$$
$$E_{t-1}(\varepsilon_t) = 0,$$
$$y_0 = 0.$$

- Two properties it has are:

$E_t(y_{t-1}) = y_t$ (Today's value is the best forecast of tomorrow's value),

$y_t = \varepsilon_1 + \varepsilon_2 + ... + \varepsilon_t$ (shocks have a permanent impact).

- A random walk is somtines referred to as ab integrated or I(1) process because we add (integrate) the ε_t's to get y_t.

Integrated GARCH (IGARCH) (Nelson 1990) Cont.

- Next consider a GARCH(1,1) with a restriction that $\alpha_1 + \beta_1 = 1$:

$$\varepsilon_t = v_t\sqrt{h_{t|t-1}}$$
$$E_{t-1}(v_t^2) = 1,\ E_{t-1}(V_t) = 0$$
$$h_{t|t-1} = \alpha_0 + \alpha_1\varepsilon_{t-1}^2 + \beta_1 h_{t-1}$$
$$\Longrightarrow E_{t-1}(h_{t+1|t}) = \alpha_0 + \alpha_1 E_{t-1}(\varepsilon_t^2) + \beta_1 h_{t|t-1}$$
$$= \alpha_0 + (\alpha_1 + \beta_1)h_{t|t-1}.$$

  Hence, $\alpha_1 + \beta_1 = 1$ implies $E_{t-1}(h_{t+1|t}) = \alpha_0 + h_{t|t-1}$.
- So, apart from α_0, the best forecast of next periods conditional variance ($h_{t+1|t}$) is last period's conditional variance ($h_{t|t-1}$).
- So, in this way the GARCH(1,1) with $\alpha_1 + \beta_1 = 1$ resembles the RW or I(1) model.

Integrated GARCH (IGARCH) (Nelson 1990) Cont.

- Therefore, Nelson (1990) refers to this as Integrated-GARCH (IGARCH) model.
- On the other hand IGRACH differs from the RW model in that shocks to the conditional variance are not permanent.
- To see this, express GARCH(1,1) as an ARCH(∞):

$$
\begin{aligned}
h_{t|t-1} &= \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 h_{t-1|t-2} = \alpha_0 + \alpha_1 L \varepsilon_t^2 + \beta_1 L h_{t|t-1} \\
&\implies (1 - \beta_1 L) h_{t|t-1} = \alpha_0 + \alpha_1 L \varepsilon_t^2 \\
&\implies h_{t|t-1} = \frac{1}{(1-\beta_1 L)} \alpha_0 + \frac{1}{(1-\beta_1 L)} \alpha_1 L \varepsilon_t^2 \\
&= \frac{\alpha_0}{1-\beta_1} + \alpha_1 \sum_{j=0^\infty} \beta_1^j L^{j+1} \varepsilon_t^2 = \frac{\alpha_0}{1-\beta_1} + \alpha_1 \sum_{j=0^\infty} \beta_1^j \varepsilon_{t-j-1}^2.
\end{aligned}
$$

- The shocks are only permanent if $\beta_1 = 1$, $\alpha_1 + \beta_1 = 1$ does not imply permanent shocks.
- Empirical appeal of IGARCH
    - GARCH(1,1) often fits financial data (e.g. stock returns) quite well.
    - Often $\widehat{\alpha}_1 + \widehat{\beta}_1 \approx 1$.

ARCH-M model (Engle, Lilien, and Robins, 1987)

- ARCH in mean - (ARCH-M) Model
- Intuition:
  - During periods of high volatility, stocks carry greater risk.
  - In order for investors to continue holding stocks during these volatile periods, this risk must be compensated by higher expected returns.
  - Put, another way, the risk premium on the stock (e.g. additional expected return to compensate the risk of holding the stock) should increase during votile periods.
  - By this logic, if our ARCH model predicts a high volatility, it should also predict a higher risk premium & therefore a higher expected return.
  - The ARCH-M models this effect by introducing the conditional variance ($h_{t|t-1}$) from the ARCH model into the model for the mean return (y_t):

$$y_t = \beta + \delta h_{t|t-1} + \varepsilon,$$

$$\left.\begin{aligned} \varepsilon = v_t\sqrt{h_{t|t-1}},\ E_{t-1}(v_t) = 0,\ \&\ E_{t-1}(v_t^2) = 1, \\ h_{t|t-1} = \alpha_0 + \sum_{i=1}^{q}\alpha_i\varepsilon_{t-i}^2, \end{aligned}\right\} \text{Standard ARCH model}$$

where $\delta h_{t|t-1}$ measures the conditional variance impacts y_t directly.

ARCH-M model (Engle, Lilien, and Robins, 1987) Cont.

- Notice that the expected return in this model is given by

$$E_t(y_{t+1}) = \beta + \delta h_{t+1|t}.$$

- If markets are efficient, then $E_t(y_{t+1}) = \beta + \delta h_{t+1|t}$ can be interpreted as a risk premium.
- For $\delta > 0$, the risk premium increases with the predicted future volatility $(h_{t+1|t})$ in accordance with financial theory.
- Notice that, unlike the other GARCH models, we've discussed so far, the GARCH-M model changes your forecast for the mean return & not just for the squared returns:

$$\widehat{y}_{t+1|t} = \widehat{\beta} + \widehat{\delta h}_{t+1|t}.$$

Threshold GARCH (TARCH)

- Stylized empirical facts: Bad news increases market volatility more than good news.
- Of course, bad news usually associated with negative returns.
- Implications: large negative returns (bad news) increase future volatility more than large positive returns (good news).
- This introduces an asymmetry into the GARCH process, whereby the past squared returns have a different impact depending on whether they represent good/bad news.
- Threshold GARCH (TGARCH) captures this by introducing a dummy variable to capture bad news (bad news dummy)

$$d_t = \begin{cases} 1, & \varepsilon_t < 0 \quad \text{(bad news)} \\ 0, & \varepsilon_t \geq 0 \quad \text{(good news or no news)} \end{cases}$$

.

- Then they interact the bad news dummy with the past squared returns together

$$h_{t|t-1} = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \lambda_1 d_{t-1} \varepsilon_{t-1}^2 + \beta_1 h_{t-1|t-2}.$$

Threshold GARCH (TARCH) Cont.

- If the news at $t-1$ is good ($\varepsilon_{t-1}^2 \geq 0$), then
  1. $d_{t-1} = 0$
  2. $h_{t|t-1} = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + \beta_1 h_{t-1|t-2}$
  3. The coefficient on ε_{t-1}^2 is α_1.
- If the news at $t-1$ is bad then
  1. $d_{t-1} = 1$
  2. $h_{t|t-1} = \alpha_0 + (\alpha_1 + \lambda_1 \varepsilon_{t-1}^2 + \beta_1 h_{t-1|t-2})$
  3. The coefficient on ε_{t-1}^2 is $\alpha_1 + \lambda_1$
- Note that if $\lambda > 0$ then
  - Good news coefficient $= \alpha_1 < \alpha_1 + \lambda_1 =$ bad news coefficient on ε_{t-1}^2.
  - So the shock (ε_{t-1}) of equal magnitude increase expected future volatility more if it is a bad news shock than if it is a good news shock.
- This canbe interpreted as a threshold model, in which the coefficients change as a certain threshold is passed: Here $\varepsilon_{t-1} = 0$ is the threshold.
- The asymmetric effect captured for this model is often referred to as the "leverage effect".

Leverage effect interpretation

- Leverage refers to the extent to which debt financing is used relative to the equity financing.
- A firm with higher leverage is one with a higher proportion of debt financing.
- Two pieces to the argument for why debt financing (or leverage) can give rise to the type of asymmetric effects captured by TARCH:
  1. Leverage makes the stock price more vulnerable:
     Why? Leverage casues the shoreholder to take a large position in the underlying assets of the firm. Thus if magnifies any change in the value of these assets.
  2. A fall in stock price (bad news) increases leverage thereby increasing future volatility.
     Why? When the stocks fall, the firms out-standing debt remains unchange but its equity level aotomatically falls with the price of the stock.

Leverage effect example: Home mortgage

- You buy a \$100 thousand dollar house using a 20% (\$ 20k) down payment and an 80% (\$ 80K) mortgage .
- To illustrate leverage effects, suppose there is a 10% decline in the value of the house in of the two subsequent years. Suppose that no principle is paid by the mortgage

Year	Home Price	%Δ Price	Debt Level	Equity in house	Return (% Δ) in equity
1 (purchase)	100	N/A	80	20	N/A
2	90	-10%	80	10	$100\%(\frac{10-20}{20}) = -50\%$
3	81	-10%	80	1	$100\%(\frac{1-10}{10}) = -90\%$

- Note how this illustrates the two pieces of the leverage argument discussed above:
  1. Leverage magnifies the volatility of the return:
     The value of the house changed by onlty 10%. The value of equity in the house changed by 50 or 90%.
  2. A negative return increases leverage & volatility.
     After the first house price drop debt financing increased from $80/100 \times 100\% = 80\%$ to $80/90 \times 100\%$.
     Consequently, the same 10% drop in price led to a 90% drop in equity instead of the previous 50% drop.

Exponential GARCH (EGARCH) (Nelson, 1991)

- Instead of modeling $h_{t|t-1}$ directly, the EGARCH model is linear model for $ln(h_{t|t-1})$:

$$ln(h_{t|t-1}) = \alpha_0 + \alpha_1(\frac{\varepsilon_{t-1}}{\sqrt{h_{t-1|t-2}}}) + \lambda_1|\frac{\varepsilon_{t-1}}{\sqrt{h_{t-1|t-2}}}| + \beta_1 ln(h_{t-1|t-2}).$$

- Note that because $x = e^{ln(x)}$, this implies an exponential model for $h_{t|t-1}$:

$$h_{t|t-1} = e^{ln(h_{t|t-1})} = exp(\alpha_0 + \alpha_1(\frac{\varepsilon_{t-1}}{\sqrt{h_{t-1|t-2}}}) + \lambda_1|\frac{\varepsilon_{t-1}}{\sqrt{h_{t-1|t-2}}}| + \beta_1 ln(h_{t-1|t-2})).$$

- Recall that $e^x = exp(x) > 0$ even for $x < 0$, in which case e^x is small, but still positive.

Exponential GARCH (EGARCH) (Nelson, 1991) Cont.

- One advantage of the EGARCH is that it gaurantees $h_{t|t-1} > 0$ without requiring coefficient restrictions:
    1. For example, we do not require $\alpha_0 > 0$ or $\alpha_1 > 0$ or $\lambda_1 > 0$, etc, in an EGARCH model.
- Similarly, it is no longer necessary to have the past $\varepsilon_t's$ enter as squared values.
- For example the term $\alpha_1(\frac{\varepsilon_{t-1}}{\sqrt{h_{t-1|t-2}}})$ can be either positive or negative depending on the sign of the ε_2. Yet either way, $h_{t|t-1}$ will be positive.
- This same term $\alpha_1(\frac{\varepsilon_{t-1}}{\sqrt{h_{t-1|t-2}}})$ is included to capture leverage effects similar to those addressed by the TARCH model.
- For example if $\alpha_1 < 0$ then bad news ($\varepsilon_{t-1} < 0$) will increase $h_{t|t-1}$ through this term.

Exponential GARCH (EGARCH) (Nelson, 1991) Cont.

- The magnitude of the news also matters:
  - It is capttured by $\lambda_1 |\frac{\varepsilon_{t-1}}{\sqrt{h_{t-1|t-2}}}|$.
  - For $\lambda_1 > 0$, larger values of $\varepsilon_{t-1}|$ (scaled by $\sqrt{h_{t-1|t-2}}$) will increase volatility more than smaller values, regardless of whether the news is good or bad.
  - This is somewhat analogous to the term ε_{t-1}^2 in the standard GARCH model.
  - However, now the "size" of the shock is measured relative to current volatility ($\sqrt{h_{t-1|t-2}}$).
  - $\frac{\varepsilon_t}{\sqrt{h_{t-1|t-2}}} = \frac{|\varepsilon_t|}{\text{standard diviation} n_{t-1}(\varepsilon_t)}$ is unit free.