Introduction to mean-varaince portfolio selection

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Empirical Financial Econometrics

Outlines

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- Mean Variance Portfolio Selection ⏩ Jump [Online Lecture]

Concepts and Notation

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- Long Horizon Returns ⏩ Jump
- Timing Convention ⏩ Jump
- Compounding Returns ⏩ Jump
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Return Definitions (without dividends)

- **Net Return (R_t)** nets (subtracts) out the purchase price:

$$\begin{aligned} R_t &= \frac{\text{Sale Price} - \text{Purchase Price}}{\text{Purchase Price}} \\ &= \frac{P_t - P_{t-1}}{P_{t-1}} \\ &= \frac{P_t}{P_{t-1}} - 1 \end{aligned}$$

- **Gross Return ($1 + R_t$)** does not net out the purchase price:

$$\begin{aligned} \text{Gross Return} &= \frac{\text{Sale price}}{\text{Purchase Price}} \\ &= \frac{P_t}{P_{t-1}} \\ &= R_t + 1 \end{aligned} \tag{1}$$

Return Definitions without dividends

- What do you mean by one- period?
  Return that would apply if we bought one period and sold the next.
- A **period** could be a minute, day, week, month, quarter, year,etc. medskip
- How often you observe the data (daily, weekly, monthly,...) is called the **frequency**
- Usually it corresponds to the sampling or observation frequency i.e: how often we observe the data

Long Horizon Returns

- **Multi periods or long-horizon returns**
- May be interested in a **return horizon**
  or **maturity** that differs from the **sampling frequency** of that data
- **e.g** Data sampled daily (daily sampling frequency)
  4 year investment horizon (interested in a 4 year return horizon)
- **e.g** Data recorded monthly (monthly sampling frequency)
  Invest in 3 month treasury bill (3 month maturity)

- **k period net-return**

$$R_t(k) = \frac{\text{sale price(sold time } t) - \text{Purchase Price(bought at time } t-k)}{\text{Purchase Price (bought at time t-k)}}$$
$$= \frac{P_t - P_{t-k}}{P_{t-k}}$$

- **k period gross-return**

$$= \frac{\text{Sale Price(t)}}{\text{Purchase Price (} t-k)}$$
$$= \frac{P_t}{P_{t-k}}$$
$$= 1 + R_t(k)$$

Timing Convention

- **Question:** why do we define $R_t(k) = \frac{P_t - P_{t-k}}{P_{t-k}}$ not $\frac{P_{t+k} - P_t}{P_t}$?

  Answer: Because we want to date our variables according to when they are first observed. We do not observe $\frac{P_{t+k} - P_t}{P_t}$ until time t+k.

- $R_t(k)$:

  t — date at which return is observed or "realized";

  k—maturity or return horizon.

- $E_t X = E[X | \textit{Available information at time } t]$

  With above dating convention anything dated at or before time t is included in this information.

Compounding Returns

Intuition- Interest on your interest. Reinvest your return from last year

$$R_t(k) = \frac{P_t - P_{t-k}}{P_{t-k}} = \frac{P_t}{P_{t-k}} - 1 \quad (2)$$

$$1 + R_t(k) = \frac{P_t}{P_{t-k}} \quad (3)$$

Suppose that $k = 2$

$$\begin{aligned} 1 + R_t(2) &= \frac{P_t}{P_{t-2}} \\ &= \left(\frac{P_{t-1}}{P_{t-1}}\right)\frac{P_t}{P_{t-2}} \\ &= \frac{P_t}{P_{t-1}}\frac{P_{t-1}}{P_{t-2}} \\ &= (1 + R_t(1))(1 + R_{t-1}(1)) \\ &= \prod_{i=0}^{1}(1 + R_{t-i}(1)) \end{aligned}$$

Compounding continued

- Solving for $R_t(2)$

$$R_t(2) = \prod_{i=0}^{1}(1 + R_{t-i}(1)) - 1$$

- Some exercise for general $R_t(k)$ gives:

$$R_t(k) = \prod_{i=0}^{k-1}(1 + R_{t-i}(1)) - 1 \tag{4}$$

Comparing Return Horizons

- **How does one- period return compare to k-period return?** Assume a constant return $R_t(1) = R$ (Same every period) and calculate $1 + R(k)$ using (4)

$$1 + R(k) = \prod_{i=0}^{k-1}(1+R) = (1+R)^k \quad (5)$$

$$R(k) = (1+R)^k - 1 \quad (6)$$

- Return **not** invariant to return horizon
- Must **Specify Horizon**, e.g Monthly return, daily return,etc
- **Convention**:If unspecified, return is yearly return.

Comparing Return Horizons (continued)

- **Question:** Bought condo for \$100, sold for \$200. Was it a good investment?

Comparing Return Horizons (continued)

- **Question:** Bought condo for \$100, sold for \$200. Was it a good investment?
- **Answer:** Can not say
  $R_t(k) = \frac{200-100}{100} = 1,$ so a one hundred percent return
  But we don't know k!
  If $k = 1$ year—— Fantastic Return
  If $k = 50$ years—— not so great

Comparing Return Horizons (continued)

- **Annualized Returns:**
- Suppose sampling frequency = yearly
- Then $R_t(1)$ = one year return
- Also suppose $R_t(1) = R$ all t
- Then solve for the one period/one year return. Using (6):

$$R(k) + 1 = (1 + R)^k \tag{7}$$

$$(1 + R) = (R(k) + 1)^{\frac{1}{k}} \tag{8}$$

$$R = (R(k) + 1)^{\frac{1}{k}} - 1 \tag{9}$$

The annualized Return

- **Question:** What constant interest rate would you have to earn for each of the next k years in order to get a return of $R(K)$ over k years.

The annualized Return

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The annualized Return

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- **Answer:** We just solved for that constant return or interest rate in (9)
- And we call this the annualized return and define it as

$$\text{Annualized}[R_t(k)] = [1 + R_t(k)]^{\frac{1}{k}} - 1 \ \text{(if t represents years)} \quad (10)$$

Example

I buy condo in 1950 for \$100,000, and sell in 2000 for \$200,000

You buy a Condo in 1990 for \$180,000 and sell in 2000 for \$200,000.

Which investment did better and by how much?

The annualized Return (continued)

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Which investment did better and by how much?

Solution Calculate actual returns, then annualized both, then compare. The details are left as an exercise.

Log Returns

- Define

$$
\begin{aligned}
r_t &= \ln(1+R_t) \\
&= \ln\left(\frac{P_t}{P_{t-1}}\right) && (11) \\
&= \ln P_t - \ln P_{t-1} && (12) \\
&= p_t - p_{t-1} && (13)
\end{aligned}
$$

where we define lower case p as the log price:

$$p_t = \ln P_t \tag{14}$$

and ln denotes the natural log

Log Returns (continued)

We define the **log long-horizon return** in the same way as:

$$
\begin{aligned}
r_t(k) &= \ln(1 + R_t(k)) && (15)\\
&= \ln\left(\frac{P_t}{P_{t-k}}\right) && (16)\\
&= \ln P_t - \ln P_{t-k} && (17)\\
&= p_t - p_{t-k} && (18)
\end{aligned}
$$

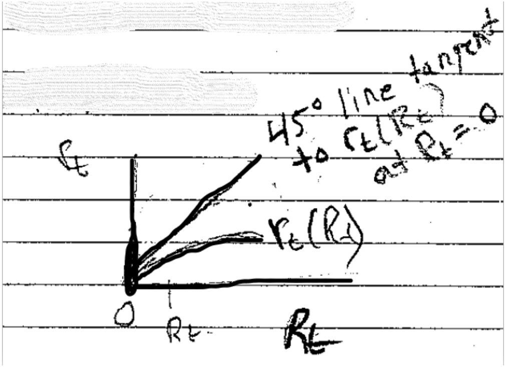
Log Returns (continued)

- Often log returns employed almost as if they are **net** returns, why? (Justified by Taylor series approximation.)
- Approx first order Taylor Expansion of r_t as a function of R_t gives:

$$
\begin{aligned}
r_t(R_t) &= \ln(1+R_t) && \text{about } R_t = 0 && (19)\\
r(R_t) &\approx r_t(0) + r_t'(0)(R_t - 0) && \text{for } R_t \approx 0 && (20)\\
&= \ln(1+0) + \left[\left. \frac{1}{1+R_t} \right|_{R_t=0} \right] R_t && && (21)\\
&= \ln(1+0) + \left[\frac{1}{1+0} \right] R_t && && (22)\\
&= R_t && && (23)
\end{aligned}
$$

where r' denotes the derivative of $r_t(R_t)$ with respect to R_t

Log Returns (Continued)



The graphic demonstrates this approximation.

$$r_t = \ln(1 + R_t) \approx R_t \qquad \text{for } R_t \approx 0 \tag{24}$$

(25)

Log Returns (Continued)

The analogous approximation for the long-horizon return is:

$$r_t(k) \quad = \quad \ln(1 + R_t(k)) \approx R_t(k) \qquad \text{for} \quad R_t(k) \approx 0 \tag{26}$$

And we can relate short and long-horizon returns by:

$$\begin{aligned}
r_t(k) &= \ln\left(1 + R_t(k)\right) && (27)\\
&= \ln\left[\prod_{i=0}^{k-1}(1 + R_{t-i}(1))\right] && (28)\\
&= \sum_{i=0}^{k-1} \ln\left(1 + R_{t-i}(1)\right) && (29)\\
r_t(k) &= \sum_{i=0}^{k-1} r_{t-i} && (30)
\end{aligned}$$

where $\prod$ and $\sum$ denote product and sum respectively

Approximate Annualized Return

- How do we annualize a log return? **Recall from** (10) **above:**

$$
\begin{aligned}
\text{Annualized}[R_t(k)] &= [1+R_t(k)]^{\frac{1}{k}} - 1 && (31)\\
1+\text{Annualized}[R_t(k)] &= [1+R_t(k)]^{\frac{1}{k}} && (32)\\
\ln\left(1+\text{Annualized}[R_t(k)]\right) &= \ln([1+R_t(k)]^{\frac{1}{k}}) && (33)\\
&= \frac{1}{k}\ln[1+R_t(k)] && (34)\\
&= \frac{r_t(k)}{k} && (35)
\end{aligned}
$$

- It turns out to be easier. We just divide by k
- Since log returns approximate net returns a quick way to approximately annualize the net return is simply to divide it by k

Relation of returns and dollars earned

The details are omitted here because we usually skip them. If you are interested you can read about them on pages 10-12 of the hand-written notes. All we need from it is the following intuitive result:

$$\text{Return} = \frac{\text{dollars earned}}{\text{dollars invested}} \tag{36}$$

Returns with dividends

- Net return includes value of dividend (say D_t)

$$\begin{aligned} R_t &= \frac{P_t - P_{t-1} + D_t}{P_{t-1}} && (37) \\ &= \frac{P_t - P_{t-1}}{P_{t-1}} + \frac{D_t}{P_{t-1}} && (38) \\ &= \text{asset appreciation} + \text{dividend yield} && (39) \\ & && (40) \end{aligned}$$

- Log return including dividends (say D_t)

$$\begin{aligned} r_t &= \ln(1 + R_t) && (41) \\ &= \ln\left(\frac{P_t + D_t}{P_{t-1}}\right) && (42) \\ &= \ln(P_t + D_t) - \ln(P_{t-1}) && (43) \end{aligned}$$

Excess Returns

- **Excess Returns**

$$Z_{it} = R_{it} - R_{0t} \tag{44}$$

  In this case, Z_{it} excess return;R_{it} return on asset i;R_{0t} reference return
- Reference return usually risk free asset.
- In practice, short-maturity (e.g 3 month) Tbill
- The excess return is payoff to arbitrage portfolio selling short asset zero and using the proceeds to buy asset i.

Excess Returns (Continued)

- **i.e**

$$\begin{aligned} \text{At } (t-1): \quad & (*) \quad \text{borrow shares of asset zero,} \\ & (*) \quad \text{sell them (to effectively borrow money)} \\ & (*) \quad \text{use proceeds to buy asset } i \\ \text{At } (t): \quad & (*) \quad \text{sell asset } i \\ & (*) \quad \text{use proceeds to buy back asset zero} \\ & (*) \quad \text{Repay your loan (in shares of asset zero)} \end{aligned}$$

Excess Returns (Continued)

- Note that is a pure **arbitrage**, because you don't actually investment your own funds; just borrow in one asset to invest in another.
- Return on this investment is either positive or negative infinity (i.e undefined)
- Why? Because

$$\text{Return} = \frac{\text{dollars earned}}{\text{dollars investment}} = \frac{\text{dollars earned}}{0}$$

  You investment nothing!
- c.f. Mortgage backed security crisis

Excess Returns (Continued)

- Calculating arbitrage pay-off, where K is the amount arbitraged

$$
\begin{aligned}
\text{Pay-off arbitrage} &= K(R_{i,t} - R_{0,t}) \\
&= KZ_{it} \\
&= (\$ \text{ amount arbitraged}) \times \text{Excess return} \quad (45)
\end{aligned}
$$

- **Note** (45) explains the importance of the concept of excess return.
- Discuss risk-return trade-off, no arbitrage concept, etc.

Log-normal distribution

- The random variable X is defined to be log-normally distributed if its natural log is normally distributed:

$$X \sim \text{log normal} \qquad \text{if } \ln(X) \sim \text{normal}$$

- In finance, it is sometimes assumed that:

$$r_t = \ln(1 + R_t) \sim N(\mu, \sigma^2)$$

- Which implies that $1 + R_{it}$ has a **log** normal distribution.

Log-normal distribution & limited liability

- Limited Liability:

$$
\begin{aligned}
1 + R_{it} = 0 \quad & \rightarrow \quad \text{Zero } \textbf{gross} \text{ return(Net return -1)} \\
& \rightarrow \quad \text{lose all assets} \\
\\
1 + R_{it} < 0 \quad & \rightarrow \quad \text{bankruptcy (previously debtor's prison)} \qquad (46)
\end{aligned}
$$

- when bankruptcy $1 + R_{it} < 0$ is ruled out then this is known as **limited liability**

Log-normal distribution & limited liability (continued)

- Log normality imposes limited liability because

$$\begin{aligned} r_{it} &= \ln(1+R_{it}) \\ \Rightarrow e^{r_{it}} &= e^{\ln(1+R_{it})} = 1 + R_{it} \\ \Rightarrow 1 + R_{it} &= e^{r_{it}} > 0 \end{aligned}$$

  Whereas assuming $R_{it} \sim (\mu, \sigma^2)$ poses the following problems:
  - Violates limited liability

Log-normal distribution & limited liability (continued)

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  - $r_{it} = \ln(1 + R_{it})$ undefined for $1 + R_{it} < 0$

Log-normal distribution & limited liability (continued)

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Whereas assuming $R_{it} \sim (\mu, \sigma^2)$ poses the following problems:

  - Violates limited liability
  - $r_{it} = \ln(1+R_{it})$ undefined for $1+R_{it} < 0$
  - So, the distribution of r_{it} not well defined unless limited liability imposed.

Limited liability (continued)

- Assuming log-normality may be too restriction for modelling returns
- Can employ other distributions that also preserve limited liability
- The easiest way to impose limited liability is to specify a distribution for for the log r_t.
- Then solving for the gross returns

$$1 + R_t = e^{r_t} > 0$$

  will always satisfy limited liability.
- **Lesson:** Model distribution of r_t rather than R_t

Solving for the mean and variance of the log-normal

We can solve for the mean and variance of the log-normal distribution using the moment generating function. If you are interested in learning more about this, then please **see pages 19-20 of the hand-written notes** and feel free to ask questions about it in office hours. **Since this is technical material we will skip it.**

Skewness

Population **Skewness** is defined by:

$$S[r] = E\left[\frac{(r-\mu)^3}{\sigma^3}\right] \qquad \text{for } \mu = E[r]$$

We estimate it by **Sample Skewness** (i.e Replace E by $\frac{1}{T}\sum_{t=1}^{T}$)

$$\hat{S} = \frac{1}{T}\sum_{t=1}^{T}\frac{(r_t-\bar{r})^3}{\hat{\sigma}^3} \qquad \bar{r} = \frac{1}{T}\sum_{t=1}^{T} r_t$$

Skewness (continued)

- Symmetric (including normal)$\Rightarrow S[R] = 0$
- $S[r] < 0 \Rightarrow$ left tail longer than right
  $\Rightarrow$ more negative than positive outliers
  e.g market crashes larger than market rallies over short time periods.
- $S[r] > 0$ right tail longer than left tail

  e.g. at individual stock level the worst a company can do is go bankrupt, but the best it can do is to be the next Facebook/Amazon/Netflix/Google (FANG). This generates a right tail.

  e.g. A small number of very wealthy individuals give a long right tail to both income and wealth distributions.

- **Kurtosis**

$$K[r] = E\left[\left(\frac{r-\mu}{\sigma}\right)^4\right]$$

- For normal distribution: $K[r] = 3$
- Excess Kurtosis $= K[r] - 3$
- Estimate by $\hat{K}$

$$\hat{K} = \frac{1}{T}\sum_{t=1}^{T}\left(\frac{r_t - \bar{r}}{\hat{\sigma}}\right)^4 = \frac{1}{T\hat{\sigma}^4}\sum_{t=1}^{T}(r_t - \bar{r})^4$$

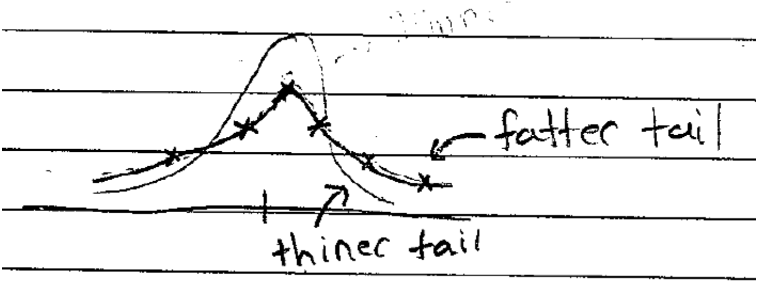
Kurtosis (continued)

Excess Kurtosis ($k > 3$) means:

- fatter tails than normal distribution
- higher probability of outliers

If 2 distributions have same variance, the one with higher Kurtosis (or fatter tails) will have a higher probability of outliers. That is of large (in absolute value) draws

e.g. Since the stock market has more large movements in both positive and negative directions than we would anticipate from the normal distribution. This explains why stock returns have excess kurtosis.



Excess Kurtosis $< 0 \rightarrow$ Opposite

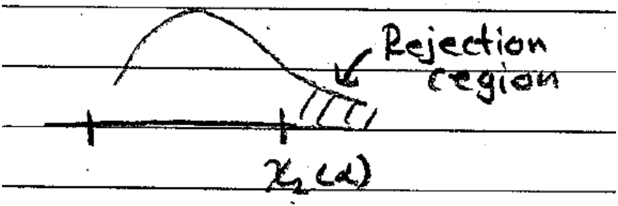
- Is normal distribution appropriate?
  (*) Is $\hat{s}[r]$ "close" to zero?
  (*) Is $\hat{K}(r)$ "close" to 3?
- **Jarque-Bera test**[1]

$$\text{JB} = \frac{T}{\sigma}\left(\hat{s}^2 + \frac{(\hat{k}-3)^2}{4}\right) \xrightarrow[H_0]{d} \chi^2_2$$

- for $H_0 : r \sim$ normally distributed
  H_A : skewness and kurtosis inconsistent with normal distribution.

[1] You are not responsible for the derivation of this result, which I do not provide.

- Reject normality if JB $> \chi_2^2(\alpha)$ for say $\alpha = 0.05$



- Note that JB becomes "large", leading us to reject when either $(\hat{s} - 0)^2$ is large or $(\hat{k} - 3)^2$ is large.
- i.e When either skewness or kurtosis inconsistent with normality.

Stylized facts	Skwness	Kurtosis
Individual stocks	$S \approx 0$ or $S > 0$	$K > 3$(Excess Kurtosis)
Stock indices	$S < 0$(left skewed)	$K > 3$(Excess Kurtosis)

- Single Risky Asset Case
- Portfolio Return
- Expected Portfolio Return
- Mean-Variance Portfolio Problem
- Sharpe Ratio

Single Risky Asset Case

- We consider the special case of 1 risky asset (no borrowing or short-sale constraints)
- **Let**

$$\begin{aligned} R_{t+1} &= \text{Return on risky asset} \\ R_{f,t+1} &= \text{Return on risk free asset} \\ w_t &= \text{Portfolio weight on risky asset} \end{aligned}$$

(i.e proportion of portfolio allocated to risky asset)

$$\begin{aligned} 1 - w_t &= \text{Portfolio weight on risk free asset} \\ \sigma_t^2 &= VAR_t(R_{t+1}) \end{aligned}$$

- **Note:** Do not assume $0 \leq w_t \leq 1$, because there are no borrowing or short sale constraints assumed.

Portfolio Return

- **Portfolio Return**

$$R_{P,t+1} = w_t R_{t+1} + (1 - w_t) R_{f,t+1} \quad (47)$$

$$= R_{f,t+1} + w_t (R_{t+1} - R_{f,t+1}) \quad (48)$$

- **What is known at time-t?**

Portfolio Return

- **Portfolio Return**

$$\begin{aligned} R_{P,t+1} &= w_t R_{t+1} + (1 - w_t) R_{f,t+1} && (47) \\ &= R_{f,t+1} + w_t (R_{t+1} - R_{f,t+1}) && (48) \end{aligned}$$

- **What is known at time-t?**
- w_t is known- has been decided
- R_{t+1} is still random
- $R_{f,t+1}$ is known because of it is risk free and if it was random it would be risky
- $R_{f,t+1}$ pays interest a time $t + 1$ but the rate is already pre-set at time t

Expected Portfolio Return

Let E_t denote the expectation conditional on information known to the investor at time t

$$\begin{aligned} R_{P,t+1} &= R_{f,t+1} + w_t(R_{t+1} - R_{f,t+1}) && (49) \\ E_t R_{P,t+1} &= E_t[\underset{C}{R_{f,t+1}} + \underset{C}{w_t}(\underset{R}{R_{t+1}} - \underset{C}{R_{f,t+1}})] && (50) \\ &= R_{f,t+1} + w_t(E_t R_{t+1} - R_{f,t+1}) && (51) \end{aligned}$$

Marking C and R below the variables equation (50) helps us remember which are treated as constants and which are treated as random inside this conditional expectation

- **Variance of Portfolio** Let VAR_t denote the variance conditional on information known to the investor at time t

$$
\begin{aligned}
R_{P,t+1} &= w_t R_{t+1} + (1 - w_t) R_{f,t+1} && (52) \\
VAR_t(R_{P,t+1}) &= VAR_t[\underset{C}{w_t} \underset{R}{R_{t+1}} + (1 - \underset{C}{w_t}) \underset{C}{R_{f,t+1}}] \\
&= VAR_t[w_t R_{t+1}] \\
&= w_t^2 VAR_t(R_{t+1}) \\
&= w_t^2 \sigma_t^2 && (53)
\end{aligned}
$$

Mean-Variance Portfolio Problem (Markowitz 1952)

- Let the investors objective function be:

$$U(w_t) = E_t(R_{P,t+1}) - \frac{k}{2} VAR_t(R_{P,t+1})$$

- Investor selects w_t to maximize $U(w_t)$

$$w_t = argmax\ U(w_t)$$

- Maximizes expected returns $(E_t[R_{P,t+1}])$, adjusted for the risk-level as captured by the variance $(Var_t(R_{P,t+1}))$

Role of Risk Aversion

- k captures risk aversion
  - $k = 0$ risk-neutral investor- maximize expected return irrespective of the risk
  - $k > 0$ risk averse
  - The larger k, the more risk averse the investor, and the larger the penalty placed on high risk(high variance) investments
  - In your project, you can obtain w_t for various values of k by estimating $E_t R_{t+1}$ and σ_t^2
  - e.g an investment advisor might have clients with different levels risk aversion and would change k accordingly
  - Since k is hard to interpret or measure precisely, a common approach is set a target portfolio expectation or variance. Either approach implicitly sets a value for k but may be more easily interpreted

Solving the mean-variance Portfolio Problem

- First we substitute our expression for the portfolio mean and variance into the investor's objective function. That is, we substitute our previous solutions for $E_t R_{P,t+1}$ and $Var_t(R_{P,t+1})$:[2]

$$U(w_t) = \underbrace{E_t(R_{P,t+1})}_{(1)} - \underbrace{\frac{k}{2} VAR_t(R_{P,t+1})}_{(2)} \quad (54)$$

$$U(w_t) = \underbrace{R_{f,t+1} + w_t(E_t R_{t+1} - R_{f,t+1})}_{(3)} - \underbrace{\frac{k}{2} w_t^2 \sigma_t^2}_{(4)} \quad (55)$$

[2] Note: We converted Term 1 into Term3 by equation (51) and Term 2 to Term 4 by equation (53)

- Next take first order condition(F.O.C) with respect to w_i

$$U(w_t) = R_{f,t+1} + w_t(E_t R_{t+1} - R_{f,t+1}) - \frac{k}{2} w_t^2 \sigma_t^2 \tag{56}$$

$$0 = \frac{\partial U(w_t)}{\partial w_t} = E_t R_{t+1} - R_{f,t+1} - 2\frac{k}{2} w_t \sigma_t^2 \tag{57}$$

- Now, solve (57) for w_t

$$k w_t \sigma_t^2 = E_t R_{t+1} - R_{f,t+1} \tag{58}$$

$$w_t = \frac{1}{k} \frac{E_t R_{t+1} - R_{f,t+1}}{\sigma_t^2} \tag{59}$$

- **Sharpe Ratio(Sharpe 1966)**

$$\text{Sharpe ratio of portfolio} = \frac{\text{Excess return}}{\text{Standard deviation}} \tag{60}$$
$$= \frac{E_t R_{p,t+1} - R_{f,t+1}}{\sqrt{VAR_t(R_{P,t+1})}} \tag{61}$$

- An unconditional version of the Sharpe ratio (where we replace conditional expectation and variance by unconditional versions)

$$\text{Unconditional Sharpe ratio} = \frac{E[R_{P,t+1} - R_{f,t+1}]}{\sqrt{VAR(R_{P,t+1})}}$$

- Which can be estimated by

$$\frac{\bar{R}_P - \bar{R}_f}{S_{RP}}$$

- Where $\bar{R}_P$ and $\bar{R}_f$ are mean portfolio and risk free returns (i,e sample means)
- And S_{RP}^2 is the sample variance of portfolio returns
- Comparing the Sharpe measures is a common way of comparing the performance of two alternative investment strategies
- In the project, you can compare Sharpe ratios for different portfolios.
- Question: What is the problem with just comparing mean returns across portfolios?