ECON 3740: INTRODUCTION TO ECONOMETRICS

INSTRUCTOR: CHAOYI CHEN
Department of Economics and Finance, University of Guelph

Lecture 5

Lecture outline

Last lecture, we learned how to derive the OLS estimator of the simple linear regression model, some meaures of fit, and how to use log technique to model the nonlineaurity (case 1). Today, we will

- continue the model in nonlinearity using the log technique (case 2 and case 3)
- focus on expected values and variances of the OLS estimators
  - expectations of the OLS estimator and homoskedasticity
  - assumptions we need to satisfy unbiasedness
  - variances of the OLS estimator and the homoskedasticity

The ordinary least squares estimator (OLS): handling non-linearity

- Case 2: *educ* is in logs (take logs of the independent variable only)

$$\hat{wage} = \hat{\beta}_0 + \hat{\beta}_1 log(educ)$$

- How do we interpret $\hat{\beta}_1$?
- Totally differentiate gives

$$\partial \hat{wage} = \hat{\beta}_1 \frac{\partial educ}{educ}$$

- To simplfy, we have

$$\underbrace{\partial \hat{wage}}_{unit\ \ change} = (\hat{\beta}_1) \underbrace{\frac{\partial educ}{educ}}_{\%\ \ change}$$

- Therefore, we can interpret the story as a 1% increase in education yields a $\hat{\beta}_1$ dollars increase in wage.

The ordinary least squares estimator (OLS): handling non-linearity

- Case 3: *wage* and *educ* are both in logs (take logs on both dependent variable and independent variable)

$$log(\hat{wage}) = \hat{\beta}_0 + \hat{\beta}_1 log(educ)$$

- How do we interpret $\hat{\beta}_1$?
- Totally differentiate gives

$$\underbrace{\frac{1}{\hat{wage}} \partial \hat{wage} \times 100}_{\%\ change} = (\hat{\beta}_1) \underbrace{\frac{\partial educ}{educ} \times 100}_{\%\ change}$$

- Therefore, we can interpret the story as a 1% increase in education yields a $\hat{\beta}_1$% increase in wage. (In log-log model, $\hat{\beta}_1$ is the elasticity of y with respect to x.)

The ordinary least squares estimator (OLS): handling non-linearity summary

- For the wage and education example,

Table: Summary of Functional Forms Involving Log: Example of Wage-Educ

Model	Dependent	Independent	Interpretation of β_1
level-level	wage	educ	1 year $\uparrow$ in educ leads to β_1 dollars $\uparrow$ in wage
log-level	log(wage)	educ	1 year $\uparrow$ in educ leads to $\beta_1\%$ $\uparrow$ in wage
level-log	wage	log(educ)	1 % $\uparrow$ in educ leads to β_1 dollars $\uparrow$ in wage
log-log	log(wage)	log(educ)	1 % $\uparrow$ in educ leads to $\beta_1\%$ $\uparrow$ in wage

Simple linear regression model: unbiasedness of OLS

- In the lecture 3, we defined the **population** model $y_i = \beta_0 + \beta_1 x_i + \mu_i$, and we claimed that the key assumption for simple regression analysis to be useful is $E(\mu|x) = 0$.
- In the lecture 4, we discussed the algebraic properties of OLS estimation.
- We now return to the population model and study the statistical properties of OLS.
- Study the statistical properties of OLS estimators means we will study properties of the distributions of $\hat{\beta}_0$, $\hat{\beta}_1$ over different random samples from the population.

Simple linear regression model: unbiasedness of OLS

- Definition: if OLS estimators are unbiased, then

$$E[\hat{\beta}_1|x] = \beta_1$$
$$E[\hat{\beta}_0|x] = \beta_0$$

- $\hat{\beta}_0$ and $\hat{\beta}_1$ are unbiased if the following four assumptions hold
  - linear in parameter: $y_i = \beta_0 + \beta_1 x_i$
  - random sample of size n. $\{(x_i, y_i) : i = 1, 2, .., n\}$
  - sample variation in the explanatory variable ($\sigma_x^2 > 0$)
  - zero conditional mean ($E(\mu|X) = 0$)

Proof for the unbiasedness:

- we have four assumptions plus

$$Y_i = \beta_0 + \beta_1 X_i + \mu_i \ , \ \bar{Y} = \beta_0 + \beta_1 X_i + \bar{\mu}$$

- We aim to get $E(\hat{\beta}_1 | x) = \beta_1$

$$E[\hat{\beta}_1] = E[\frac{\sum_{i=1}^{n} (x_i - \bar{x}_i)(y_i - \bar{y}_i)}{\sum_{i=1}^{n} (x_i - \bar{x}_i)(x_i - \bar{x}_i)}]$$

- substitute for Y_i and $\bar{Y}$ with $\beta_0 + \beta_1 X_i + \bar{\mu}$ and $\beta_0 + \beta_1 X_i + \bar{\mu}$

$$= E[\frac{\sum_{i=1}^{n} (x_i - \bar{x}_i)(\beta_0 + \beta_1 X_i + \bar{\mu} - (\beta_0 + \beta_1 X_i + \bar{\mu}))}{\sum_{i=1}^{n} (x_i - \bar{x}_i)(x_i - \bar{x}_i)}]$$

- Note that β_0 cancel out

$$= E[\frac{\sum_{i=1}^{n} (x_i - \bar{x}_i)(\beta_1 (X_i - \bar{X}) + (\mu_i - \bar{\mu}))}{\sum_{i=1}^{n} (x_i - \bar{x}_i)(x_i - \bar{x}_i)}]$$

Simple linear regression model: unbiasedness of OLS

- rewrite & use expectation rule, we have

$$= E[\frac{\beta_1 \sum_{i=1}^{n} \left(x_i - \bar{x}_i\right)\left(x_i - \bar{x}_i\right)}{\sum_{i=1}^{n} \left(x_i - \bar{x}_i\right)\left(x_i - \bar{x}_i\right)}] + E[\frac{\sum_{i=1}^{n} \left(x_i - \bar{x}_i\right)\left(\mu_i - \bar{\mu}\right)}{\sum_{i=1}^{n} \left(x_i - \bar{x}_i\right)\left(x_i - \bar{x}_i\right)}]$$

- Note that $\sum_{i=1}^{n}(x_i - \bar{x})(\mu_i - \bar{\mu}) = \sum_{i=1}^{n}(x_i - \bar{x})\mu_i$ (will show it in lecture)
- Therefore,

$$E[\hat{\beta}_1] = \beta_1 \times \underbrace{E[\frac{\sum_{i=1}^{n} \left(x_i - \bar{x}_i\right)\left(x_i - \bar{x}_i\right)}{\sum_{i=1}^{n} \left(x_i - \bar{x}_i\right)\left(x_i - \bar{x}_i\right)}]}_{=1} + E[\frac{\sum_{i=1}^{n} \left(x_i - \bar{x}_i\right)\mu_i}{\sum_{i=1}^{n} \left(x_i - \bar{x}_i\right)\left(x_i - \bar{x}_i\right)}]$$

- By the law of iterated expectation (LIE)

$$= \beta_1 + E[\frac{\sum_{i=1}^{n} \left(x_i - \bar{x}_i\right)\mu_i | x_i}{\sum_{i=1}^{n} \left(x_i - \bar{x}_i\right)\left(x_i - \bar{x}_i\right)}]$$

- Hence, $E[\hat{\beta}_1] = \beta_1$ if $E[\mu_i | x_i] = 0$, which is just our zero conditional mean assumption. This completes our proof.

Simple linear regression model: variance of OLS

- We have discussed the mean of the OLS estimator and derived its unbiasedness.
- We are also interested to see the variance of the OLS estimator ($Var(\beta_1)$). Why? Intuitively, the unbiasedness shows the sampling distribution of $\hat{\beta_1}$ is centered about β_1. However, it is important to know how far we can expect $\hat{\beta_1}$ to be away from β_1 on avergae.
- Under assumptions 1-4, the variance of the OLS estimators can be computed but with very complicated expression. To simplify the problem, we need one more assumption, which is traditional for cross-sectional analysis, "the homoskedasticity" assumption.

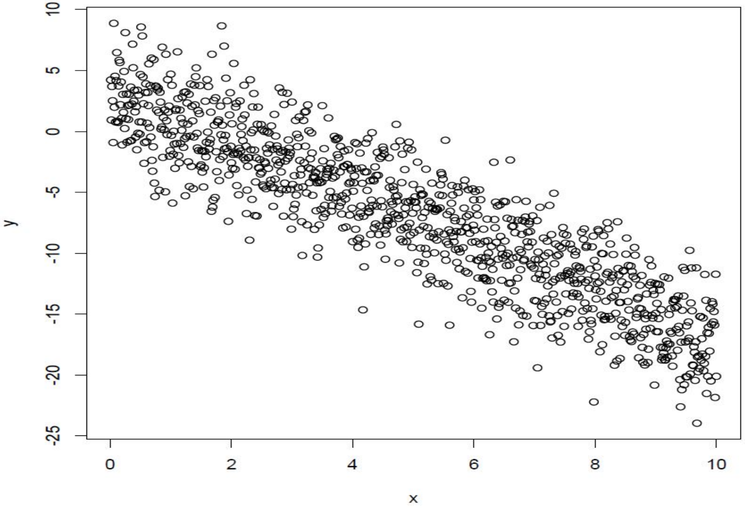
Simple linear regression model: variance of OLS

- *Homoskedasticity*: The error μ has the **same** variance given any value of the explanatory variable. In other words,

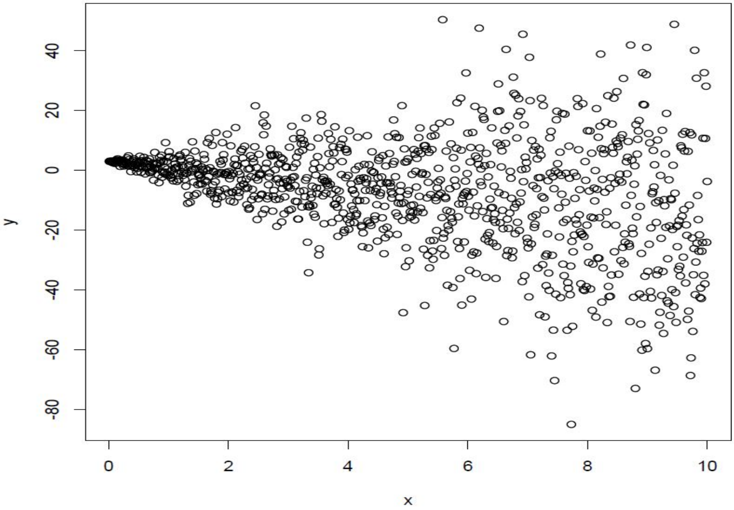
$$Var(\mu|x) = \sigma^2$$

- variance of error is common across x
- assumptions 1-4 plus "the homoskedasticity" assumption constitute the famous "Gauss-Markov Assumptions".
- since $Var(\mu|x) = E(\mu^2|x) - [E(\mu|x)]^2$ (why?) and $E(\mu|x) = 0$, $E(\mu^2|x) = \sigma^2$, $Var(\mu) = \sigma^2$. That is, σ^2 is both conditional and unconditional variance for μ.
- if $Var(\mu|x) \neq Var(\mu)$, errors are *heteroskedastic*. The error variance changes as x changes.

Simple linear regression model: homoskedastic error - a graph illustration



Simple linear regression model: graph for heteroskedastic error - a graph illustration



Simple linear regression model: sample variance of the OLS estimators

- Under assumptions 1-5, we can show the variance of $\hat{\beta}_1$

$$Var(\hat{\beta}_1) = \frac{\sigma^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

- Under assumptions 1-5, we can show the variance of $\hat{\beta}_0$

$$Var(\hat{\beta}_1) = \frac{\sigma^2 n^{-1} \sum_{i=1}^{n} x_i^2}{\sum_{i=1}^{n}(x_i - \bar{x})^2}$$

- We will go through the derivations next lecture.