Empirical Panel Data: Lecture 6

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Topic 3: A review on the random effect model and the GLS estimator

$$\underset{(T\times 1)}{\boldsymbol{y}_i} = \underset{(T\times k+1)}{\boldsymbol{X}_i}\underset{(k+1\times 1)}{\gamma} + \underset{(T\times 1)}{\boldsymbol{u_i}},$$
$$\underset{(T\times 1)}{\boldsymbol{u_i}} = \underset{(T\times 1)}{\boldsymbol{e}}\alpha_i + \underset{(T\times 1)}{\boldsymbol{\varepsilon_i}}$$

- $\underset{(k+1\times 1)}{\boldsymbol{X}_i} = \left(\underset{(T\times 1)}{\boldsymbol{e}}, \underset{(T\times k)}{\boldsymbol{x_i}}\right) \underset{(k+1\times 1)}{\gamma} = \left(\mu, \boldsymbol{\beta}^\top\right)^\top$

  $$\Omega = E(\boldsymbol{u_i}\boldsymbol{u_i}^\top) = E\left[\left(\boldsymbol{e}\alpha_i + \boldsymbol{\varepsilon_i}\right)\left(\boldsymbol{e}\alpha_i + \boldsymbol{\varepsilon_i}\right)^\top\right] = \sigma_\alpha^2 \boldsymbol{e}\boldsymbol{e}^\top + \sigma_\varepsilon^2 I_T$$

- $$\widehat{\gamma}^{GLS} = \left(\sum_{i=1}^n \boldsymbol{X_i}^\top \Omega^{-1} \boldsymbol{X_i}\right)^{-1} \left(\sum_{i=1}^n \boldsymbol{X_i}^\top \Omega^{-1} \boldsymbol{y_i}\right).$$

- Under Assumptions RE, $\widehat{\gamma}^{GLS}$ is BLUE estimator!

Topic 3: Another form of the inverse of the variance-covariance matrix

- Note that we can rewrite Ω^{-1} as

$$\Omega^{-1} = \frac{1}{\sigma_\varepsilon^2}\left(Q + \psi \frac{1}{T} ee^\top\right)$$

- $Q = \left(I_T - ee^\top / T\right)$ and $\psi = \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + T\sigma_\alpha^2}$
- Therefore, given Ω^{-1}, we have

$$\begin{bmatrix} \widehat{\mu}^{GLS} \\ \widehat{\beta}^{GLS} \end{bmatrix} = \begin{bmatrix} \psi n T & \psi T \sum_{i=1}^n \bar{\boldsymbol{x}}_i \\ \psi T \sum_{i=1}^n \bar{\boldsymbol{x}}_i & \sum_{i=1}^n \boldsymbol{X_i}^\top Q \boldsymbol{X_i} + \psi T \sum_{i=1}^n \bar{\boldsymbol{x}}_i \bar{\boldsymbol{x}}_i^\top \end{bmatrix}^{-1}$$

$$\times \begin{bmatrix} \psi n T \bar{\boldsymbol{y}} \\ \sum_{i=1}^n \boldsymbol{X_i}^\top Q \boldsymbol{X_i} + \psi^\top T \sum_{i=1}^n \bar{\boldsymbol{x}}_i \bar{\boldsymbol{y}} \end{bmatrix}$$

Topic 3: GLS as a weighted estimator of the LSDV and between estimator

- The between-group estimator or **between estimator** $\widehat{\beta}^{BE}$ is the OLS estimator obtained in the model $\bar{\boldsymbol{y}}_{\boldsymbol{i}} = c\bar{\boldsymbol{x}}_{\boldsymbol{i}}\beta + \boldsymbol{u}_{\boldsymbol{i}}$:

$$\Longrightarrow \widehat{\beta}^{BE} = \left(\sum_{i=1}^{n} \left(\bar{\boldsymbol{x}}_{\boldsymbol{i}} - \bar{\boldsymbol{x}}\right)\left(\bar{\boldsymbol{x}}_{\boldsymbol{i}} - \bar{\boldsymbol{x}}\right)^{\top}\right)^{-1} \left(\sum_{i=1}^{n} \left(\bar{\boldsymbol{x}}_{\boldsymbol{i}} - \bar{\boldsymbol{x}}\right)\left(\bar{\boldsymbol{y}}_{\boldsymbol{i}} - \bar{\boldsymbol{y}}\right)\right)$$

- $\widehat{\beta}^{BE}$ is called the between-group estimator because it **ignores** variation within the group.

$$\text{Recall: } \widehat{\beta}^{GLS} = \left(\frac{1}{T}\sum_{i=1}^{n} \boldsymbol{X}_{\boldsymbol{i}}^{\top} Q \boldsymbol{X}_{\boldsymbol{i}} + \psi \sum_{i=1}^{n} \left(\bar{\boldsymbol{x}}_{\boldsymbol{i}} - \bar{\boldsymbol{x}}\right)\left(\bar{\boldsymbol{x}}_{\boldsymbol{i}} - \bar{\boldsymbol{x}}\right)^{\top}\right)^{-1}$$

$$\times \left(\frac{1}{T}\sum_{i=1}^{n} \boldsymbol{X}_{\boldsymbol{i}}^{\top} Q \boldsymbol{y}_{\boldsymbol{i}} + \psi \sum_{i=1}^{n} \left(\bar{\boldsymbol{x}}_{\boldsymbol{i}} - \bar{\boldsymbol{x}}\right)\left(\bar{\boldsymbol{y}}_{\boldsymbol{i}} - \bar{\boldsymbol{y}}\right)\right)$$

- $\widehat{\beta}^{GLS}$ is a weighted average of $\widehat{\beta}^{LSDV}$ and $\widehat{\beta}^{BE}$!

Topic 3: LSDV, Between estimator and GLS estimator

- If $\psi \longrightarrow 0$, GLS estimator converges to LSDV (within-group) estimator

$$\widehat{\beta}^{GLS} \xrightarrow[\psi \longrightarrow 0]{p} \widehat{\beta}^{LSDV}$$

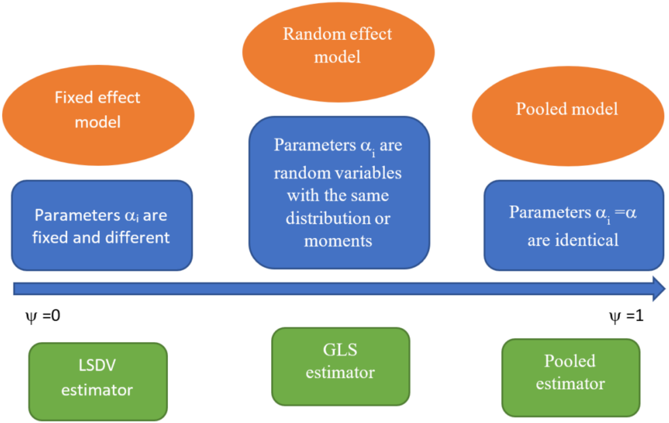
- If $\psi \longrightarrow 1$, GLS estimator converges to pooled OLS estimator

$$\widehat{\beta}^{GLS} \xrightarrow[\psi \longrightarrow 1]{p} \widehat{\beta}^{pooled}$$

- Intuition: The parameter $\psi = \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + T\sigma_\alpha^2}$ measures the weight given to the between-group variation.
  - This variation is completely ignored in the LSDV (or fixed-effects model) procedure ($\psi = 0$).
  - In the OLS procedure (pooled model), between-group variation is considered a full weight $\psi = 1$.

Topic 3: A new view on random effect

- A fun fact: treating α_i as random coefficients provide an intermediate solution between treating them all as different (fixed effects, LSDV) and treating them all as equal (pooled model).



Topic 3: LSDV and GLS more interpretation

- As $\psi = \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + T\sigma_\alpha^2}$, we have $\lim_{T \longrightarrow \infty} \psi = 0$
- Therefore, when $T \longrightarrow \infty$, the GLS estimator converges to the LSDV estimator !
- Interpretation:
  1. When $T \longrightarrow \infty$, we have an infinite number of observations for each i.
  2. Therefore, we can consider each α_i as a random variable which has been drawn once and forever
  3. For each i, we assume that they are just like fixed parameters

Topic 3: Asymptotic variance-covariance matrix for GLS

- Under Assumptions RE, the **asymptotic** variance-covariance matrix of the GLS estimator is given by:

$$\text{Var}(\widehat{\beta}^{GLS}) = \sigma_\varepsilon^2 \left(\sum_{i=1}^{n} \boldsymbol{X_i}^\top Q \boldsymbol{X_i} + \psi T \sum_{i=1}^{n} (\bar{\boldsymbol{x}}_{\boldsymbol{i}} - \bar{\boldsymbol{x}}) (\bar{\boldsymbol{x}}_{\boldsymbol{i}} - \bar{\boldsymbol{x}})^\top \right)^{-1}$$

- Remark: Note that $\text{Var}(\widehat{\beta}^{LSDV}) = \sigma_\varepsilon^2 \left(\sum_{i=1}^{n} \boldsymbol{X_i}^\top Q \boldsymbol{X_i} \right)^{-1}$. As ψ is positive by definition, $\text{Var}(\widehat{\beta}^{LSDV}) - \text{Var}(\widehat{\beta}^{GLS})$ is a positive semidefinite matrix.
  $\Longrightarrow$ LSDV is not BLUE!

Topic 3: Feasible GLS

- Given σ_ε^2 and σ_α^2, we can estimate the random effect model by GLS.
- However, in reality, they are unknown. In this case, we can use a two-step GLS estimation procedure - called feasible GLS.
    1. In the first step, we estimate the variance components using some consistent estimators.
    2. In the second step, we substitute their estimated values into

$$\widehat{\gamma}^{GLS} = \left(\sum_{i=1}^{n} \boldsymbol{X_i}^\top \widehat{\Omega}^{-1} \boldsymbol{X_i} \right)^{-1} \left(\sum_{i=1}^{n} \boldsymbol{X_i}^\top \widehat{\Omega}^{-1} \boldsymbol{y_i} \right).$$

Topic 3: σ_ε^2 and σ_α^2 estimators

- We can use the within and between-group residuals to estimate the first-step σ_ε^2 and σ_α^2 as

$$\sigma_\varepsilon^2 = \frac{\sum_{i=1}^n \sum_{t=1}^T \left((y_{it} - \bar{\boldsymbol{y}}_{\boldsymbol{i}}) - (x_{it} - \bar{\boldsymbol{x}}_{\boldsymbol{i}}) \widehat{\beta}^{LSDV} \right)^2}{n(T-1) - K}$$

$$\widehat{\sigma}_\alpha^2 = \frac{\sum_{i=1}^n \left(\bar{\boldsymbol{y}}_{\boldsymbol{i}} - \widehat{\mu}^{BE} - \bar{\boldsymbol{x}}_{\boldsymbol{i}} \widehat{\beta}^{BE} \right)^2}{n - K - 1} - \widehat{\sigma}_\varepsilon^2$$

- Then, we can compute the estimate of ψ as $\widehat{\psi} = \frac{\widehat{\sigma}_\varepsilon^2}{\widehat{\sigma}_\varepsilon^2 + T\widehat{\sigma}_\alpha^2}$ and the estimate of Ω inverse as

$$\widehat{\Omega}^{-1} = \frac{1}{\widehat{\sigma}_\varepsilon^2} \left(Q + \widehat{\psi} \frac{1}{T} \boldsymbol{e}\boldsymbol{e}^\top \right)$$

Topic 3: Remarks on feasible GLS

- Remark on large sample: When the sample size is large (either $n \longrightarrow \infty$, or $T \longrightarrow \infty$, or both), the two-step GLS estimator will have the same asymptotic efficiency as the GLS procedure with known variance components.
- Remark on small sample: Even for moderate sample size (for $T \geq 3$, $n - (K+1) \geq 9$ or for $T > 2$, , $n - (K+1) \geq 10$), the two-step procedure is still more efficient than the LSDV estimator in the sense that the difference between the covariance matrices of the covariance estimator and the two-step estimator is nonnegative definite

- `xtreg lrgdpnagrowth lccongrowth lckgrowth lpopgrowth, re robust`

```
Random-effects GLS regression                   Number of obs     =      9,229
Group variable: country1                        Number of groups  =        180

R-sq:                                           Obs per group:
     within  = 0.2911                                         min =         24
     between = 0.6515                                         avg =       51.3
     overall = 0.3089                                         max =         64

                                                Wald chi2(3)      =     145.53
corr(u_i, X)   = 0 (assumed)                    Prob > chi2       =     0.0000

                             (Std. Err. adjusted for 180 clusters in country1)
------------------------------------------------------------------------------
             |               Robust
lrgdpna_gr~h |      Coef.   Std. Err.      z    P>|z|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
lccon_growth |   .3837601   .0392738     9.77   0.000     .3067848    .4607354
  lck_growth |   .0704687   .0323552     2.18   0.029     .0070537    .1338838
 lpop_growth |   .4287039    .083819     5.11   0.000     .2644218    .5929861
       _cons |   1.024606   .2404972     4.26   0.000     .5532401    1.495972
-------------+----------------------------------------------------------------
     sigma_u |  .40670155
     sigma_e |  5.4526324
         rho |  .00553261   (fraction of variance due to u_i)
------------------------------------------------------------------------------
```