Model diagnostics/selection in GARCH framework

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Empirical Financial Econometrics

Outlines

- Model diagnostics/selection in GARCH framework Jump [Online Lecture]

Model diagnostics/selection in GARCH framework

- Plots of the data
- Box Ljung test
- Lagrange test
- Model selection criterion
- Residual diagnostics
- Test for leverage effects

Plots of the data

- Time plots of returns often show clear indications of conditional volatility with the (absolute value of) returns: tending to be much larger in some period than others ⏩ Plot
- ACFs of squared residuals
  1. If there are GARCH effects present then we should expect ε_t^2 to be corrected with past squared errors $(\varepsilon_{t-1}^2, \varepsilon_{t-2}^2, ...)$
  2. Suggests plotting an ACF of ε_t^2
  3. In practice don't observe ε_t^2 directly. So plot ACF pf $\widehat{\varepsilon}_t^2$
  4. This means that we need to have decided on a model for the mean return (e.g. intercept plus noise or ARMA) before we construct the ACF of $\widehat{\varepsilon}_t^2$

Plots of the data, ACFs of squared residuals

- To recap, at this period
  - We have selected & estimated our mean model (e.g. ARMA)
  - And this gives us $\widehat{\varepsilon}_t^2$
  - But we have not yet selected a model for volatility
  - We simply look at the ACF of $\widehat{\varepsilon}_t^2$ for indications of volatility clustering.
- Step 1: Select and estimated model for mean of $y_t \implies$ obtain $\widehat{\varepsilon}_t^2$
- Step 2: Estimate the unconditional varaince $\widehat{\sigma}^2 = \frac{1}{T}\sum_{t=1}^{T}\widehat{\varepsilon}_t^2$
  estimates $\sigma^2 = E[\varepsilon_t^2]$
  Note: used to center ε_t^2 about its unconditional mean:
  $E[\varepsilon_t^2] = \sigma^2 \implies E[\varepsilon_t^2 - \sigma^2] = 0$

Plots of the data, ACFs of squared residuals Cont.

- Step 3: Estimate sample correlation for ε_t^2, $\rho_i = cor(\varepsilon_t^2, \varepsilon_{t-i}^2)$ (unconditional correlation), by

$$\widehat{\rho}_i = \widehat{cor}(\varepsilon_t^2, \varepsilon_{t-i}^2) = \frac{\widehat{COV}(\varepsilon_t^2, \varepsilon_{t-i}^2)}{\widehat{Var}(\varepsilon_t^2)}$$

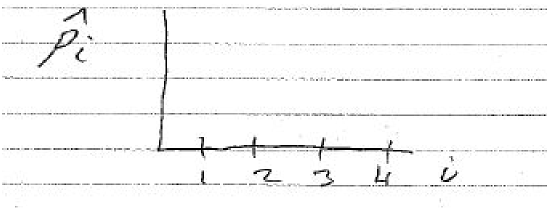
$$= \frac{\widehat{E}\left\{\left(\varepsilon_t^2 - \widehat{E}[\varepsilon_t^2]\right)\left(\varepsilon_{t-i}^2 - \widehat{E}[\varepsilon_{t-i}^2]\right)\right\}}{\widehat{E}\left\{\left(\varepsilon_t^2 - \widehat{E}[\varepsilon_t^2]\right)^2\right\}} \text{ (Note that } E[\varepsilon_t^2] = E[\varepsilon_{t-i}^2] = \sigma^2\text{)}$$

$$= \frac{\widehat{E}\left\{\left(\varepsilon_t^2 - \widehat{\sigma}^2]\right)\left(\varepsilon_{t-i}^2 - \widehat{\sigma}^2]\right)\right\}}{\widehat{E}\left\{\left(\varepsilon_t^2 - \widehat{\sigma}^2]\right)^2\right\}}$$

$$\boxed{\widehat{\rho}_i = \frac{\frac{1}{T}\sum_{t=i+1}^{T}\left(\widehat{\varepsilon}_t^2 - \widehat{\sigma}^2\right)\left(\widehat{\varepsilon}_{t-i}^2 - \widehat{\sigma}^2\right)}{\frac{1}{T}\sum_{t=1}^{T}\left(\widehat{\varepsilon}_t^2 - \widehat{\sigma}^2\right)^2}}$$

Plots of the data, ACFs of squared residuals Cont.

- Step 4: Plot $\widehat{\rho}_i$ against i just like for original ACF.



- Step 5: Add (pointwise) two standard error bands around zero of $0 \pm \frac{2}{\sqrt{T}}$. similar to original ACF.
- Step 6: Perform Box-Ljung test to determine if there is conditional volatility (see below).

Formal test for conditional volatility - Box Ljung test

- Box - Ljung Test
  - Similar to use of Box-ljung test for serial in ε_t - but now applied to ε_t^2.
  - Test that the first k autocorrelations of ε_t^2 are zero

$$H_0: \ \rho_1 = \rho_2 = ... = \rho_k = 0$$
$$H_A: \ \text{Not } H_0 \ (\text{At least one } \rho \neq 0)$$

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  - Under H_0 there may not be ARCH or GARCH effects.
  - If reject H_0 need to consider GARCH type models to capture the volatility dynamics

$$Q = T(T+2)\sum_{i=1}^{k}\frac{\widehat{\rho}_i^2}{T-i} \sim \chi_k^2 \text{ Under the null for large T.}$$

Reject at significance level α figure

Formal test for conditional volatility - Lagrange test

- Lagrange multiplier test for the ARCH disturbances (Engle 1982)
  - Based on regression of $\widehat{\varepsilon}_t^2$ on q lags

$$\widehat{\varepsilon}_t^2 = \alpha_0 + \alpha_1 \widehat{\varepsilon}_{t-1}^2 + \alpha_2 \widehat{\varepsilon}_t^2 + \ldots + \alpha_q \widehat{\varepsilon}_{t-q}^2 + \text{error}$$

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  - If there are no ARCH effects, then the coefficients on all lags should be zero

$$H_0: \ \alpha_1 = \alpha_2 = \ldots = \alpha_p = 0 \text{ No ARCH effects}$$
$$H_A: \ \text{Not } H_0. \text{ ARCH effects present}$$

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  - Two Possible Tests
    1. Standard F-test
    2. Lagrange Multiplier Test

$$TR^2 \sim \chi_q^2 \ (\text{Under the} H_0, \ \text{T large}),$$

where R^2 is the R^2 statistic.
Reject at size α is $TR^2 > \chi_q^2(\alpha)$ figure

Model selection criterion

- We derived the model selection criterion for ARCH & GARCH in the notes labelled "ML-GARCH".
- They are based on the value of the log-likelihood penalized for the size of the model.
- The basic idea is still the same: select the model(s) with the best penalized fit - here fit being measured by the log-likelihood.
- Again we may not wish to perform additional diagnostics on models that are not much worse than the best model in terms of penalized fit.
- Usually your software will provide the AIC & BIC for each model.
- Normally calculated for a range of GARCH(p,q) models for all combinations of p and q satisfying $0 \leq p \leq \bar{p}$, $0 \leq q \leq \bar{q}$, where $\bar{p}$ and $\bar{q}$ are upper limits you set.
- Be sure to read the command description in your software to see exactly what is being calculated
  - For example, should you select model with highest or lowest AIC/SBC
  - Also make sure your AIC/SBC calculations consistent accross models.

Residual diagnostics

- After estimating one or more plausible candidates GARCH models, we can apply "residual diagnostics" to see if we have captured all of the GARCH effects.
- Basic idea is: If we have a "good" GARCH specification there should be no GARCH effects in the "residual" from the GARCH model.
- But, the GARCH model is a model of the residual, so what do we mean by a "residual" from the GARCH model?
- Recall

$$\varepsilon_t = v_t \sqrt{h_{t|t-1}} \ , \ E_{t-1} v_t = 0 \ , \ E_{t-1} v_t^2 = 1$$

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- Note that $E_{t-1} v_t^2 = 1 \Longrightarrow cov(v_t^2, v_{t-1}^2) = 0$ for $i \neq 0$ by a law of iterated expectation argument (please try this at home as practice).
- So v_t^2 is serially uncorrelated
- And $v_t^2 = \frac{\varepsilon_t^2}{h_t|t-1}$.

Residual diagnostics Cont.

- If we have a good GARCH specification then $\widehat{v}_t^2 = \frac{\widehat{\varepsilon}_t^2}{\widehat{h}_{t|t-1}} \approx \frac{\varepsilon_t^2}{h_{t|t-1}} = v_t^2$. So that $\widehat{\varepsilon}_t^2$ should be approximately uncorrelated.
- Calculate ACF for $\widehat{v}_t^2$ using

$$\widehat{\rho}_i = \frac{\frac{1}{T}\sum_{t=i+1}^{T}\left(\widehat{v}_t^2 - \widehat{\sigma}_v^2\right)\left(\widehat{v}_{t-i}^2 - \widehat{\sigma}_v^2\right)}{\frac{1}{T}\sum_{t=1}^{T}\left(\widehat{v}_t^2 - \widehat{\sigma}_v^2\right)^2}$$

- And the large sample two standard error bands about zero: $0 \pm \frac{2}{\sqrt{T}}$.
- Most of the $\widehat{\rho}_i$ should fall inside the two standard error bands.
- A formal test can be done using the Box-Ljung statistic, adjusting the degrees of freedom for the number of GARCH parameters estimated

$$H_0: \ \rho_1 = \rho_2 = ... = \rho_k = 0 \,(\widehat{v}_t^2 \text{uncorrelated} \longrightarrow \text{good GARCH specification})$$
$$H_A: \ \text{Not } H_0 \ (\text{at least one} \rho_i \neq= 0)$$

$$Q = T(T+2)\sum_{i=1}^{k}\frac{\widehat{\rho}_i^2}{(T-i)} \sim \chi^2_{k-h},$$

where $h = p + q + 1$ is the number of GARCH parameters estimated.

Residual diagnostics Cont.

- If we reject H_0 then our model has not adequately captured the GARCH type effects that give rise to serial correlation in $\widehat{\varepsilon}_t^2$.
- We can also use the F or lagrange multiplier test for ARCH effects on the standradized residuals $\widehat{v}_t$

$$\widehat{v}_t^2 = \alpha_0 + \alpha_1 \widehat{v}_{t-1}^2 + \alpha_2 \widehat{v}_{t-2}^2 + \ldots + \alpha_q \widehat{v}_{t-q}^2 + error$$

$$H_0: \ \alpha_1 = \alpha_2 = \ldots = \alpha_q = 0 \ (\text{No remaining ARCH effects})$$

$$H_A: \ \text{Not } H_0$$

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Test for leverage effects

- Idea: Test the standardize residuals of your fitted GARCH-type models to see if these are remaining leverage effects (asymmetries).
- If these are leverage effects, you may consider a TARCH or EGARCH type or adding the threshold term of TARCH to the model you already have in place to capture the leverage.
- Test 1: Regress squared standardized residuals on its lag-levels
  - $$\underbrace{\frac{\widehat{\varepsilon}_t^2}{\underset{\substack{\uparrow \\ squared}}{\widehat{h}_{t|t-1}}}} = \alpha_0 + \alpha_1 \frac{\widehat{\varepsilon}_{t-1}^2}{\underset{\substack{\uparrow \\ \text{Not} \\ \text{squared}}}{\sqrt{\widehat{h}_{t-1|t-2}}}} + \alpha_2 \frac{\widehat{\varepsilon}_{t-2}^2}{\sqrt{\widehat{h}_{t-2|t-3}}} + \ldots + \alpha_h \frac{\widehat{\varepsilon}_{t-h}^2}{\sqrt{\widehat{h}_{t-h|t-h-1}}} + error$$

    $$\widehat{v}_t^2 = \alpha_0 + \alpha_1 \widehat{v}_{t-1} + \alpha_2 \widehat{v}_{t-2} + \ldots + \alpha_h \widehat{v}_{t-h} + error$$

    $$H_0: \ \alpha_1 = \alpha_2 = \ldots = \alpha_h = 0 \text{ (No leverage effects left in standardized residuals)}$$

    $$H_A: \ \text{Not } H_0$$

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  - F-test can be used.
  - Reject $H_0 \longrightarrow$ need to (further) model leverage effects

Test for leverage effects: test 2

- Test 2: Test for sign bias in squared standardized residuals (Engle & Ng, 1993)
  - $$d_{t-1} = \begin{cases} 1, \ \widehat{\varepsilon}_{t-1} < 0 \text{ ("Bad news dummy")}, \\ 0, \ \widehat{\varepsilon}_{t-1} > 0 \end{cases}.$$
    $$\underbrace{\frac{\widehat{\varepsilon}_t^2}{\widehat{h}_{t|t-1}}} = \alpha_0 + \alpha_1 d_{t-1} + error.$$
    $$\widehat{v}_t^2 = \alpha_0 + \alpha_1 d_{t-1} + error.$$
    $$H_0: \ \alpha_1 = 0 \text{ (No leverage effects left in standardized residuals)},$$
    $$H_A: \ \alpha_1 \neq 0.$$
  - Intuition: $\alpha_1 \neq 0 \implies$ standardized residuals can be predicted based on good and bad news $\implies$ leverage effects remain.

Test for leverage effects: test 3

- Test 3: Combined test: Regress squared standardized residuals on both the sign of the lagged residuals (d_{t-1}) and its standardized lags interacted with its sign
- For example:

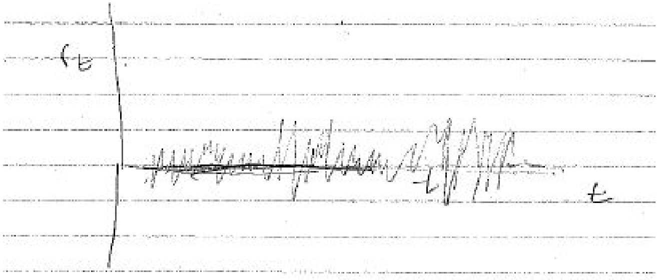
$$\frac{\hat{\varepsilon}_t^2}{\hat{h}_{t|t-1}} = \alpha_0 + \alpha_1 d_{t-1} + \alpha_2 d_{t-1} \left(\frac{\hat{\varepsilon}_{t-1}}{\sqrt{\hat{h}_{t-1|t-2}}} \right) + \alpha_3 \left(\frac{\hat{\varepsilon}_{t-1}}{\sqrt{\hat{h}_{t-1|t-2}}} \right) + error.$$

which can also be rewritten as

$$\hat{v}_t^2 = \alpha_0 + \alpha_1 d_{t-1} + \alpha_2 d_{t-1} \hat{v}_t + \alpha_3 \hat{v}_{t-1} + error.$$

- The null typothesis of no leverage effects is then

$$H_0: \; \alpha_1 = \alpha_2 \alpha_3 = 0.$$



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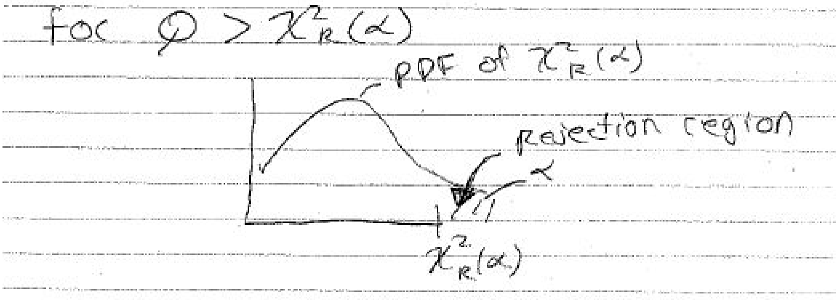
for Q > $\chi^2_R(\alpha)$
PDF of $\chi^2_R(\alpha)$
Rejection region
α
$\chi^2_R(\alpha)$

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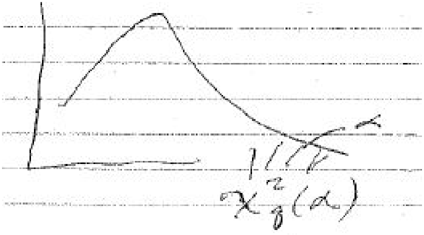

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