Lab 2 Solutions (PARTS)

January 30, 2017

1 Chapter 5.6

Question: Suppose that a firm's total product function is $y = 40L^2$ - L^3. Show that the average product of labor, $AP(L)$, rises when marginal product of labor, $MP(L)$, exceeds $AP(L)$, falls when $MP(L)$ is less than $AP(L)$, and is horizontal at the point where $MP(L) = AP(L)$.

Hints: At $MP(L) = AP(L)$, we can derive that:

$$80L - 3L^2 = 40L - L^2 \tag{1}$$

$$L = 20 \tag{2}$$

Therefore, we have:

$$\begin{cases} if\ L < 20, \ \rightarrow MP(L) \ > \ AP(L) \\ if\ L = 20, \ \rightarrow MP(L) \ = \ AP(L) \\ if\ L > 20, \ \rightarrow MP(L) \ < \ AP(L) \end{cases} \tag{3}$$

Now, let's find the derivative of $AP(L)$ (we should determine the sign of the derivative at each labor level to inference the increasing, decreasing and horizontal intervals):

Differentiate $AP(L)$ w.r.t L, we have:

$$FOC : \frac{dAP(L)}{dL} = 40 - 2L = 0 \Rightarrow L = 20 \tag{4}$$

$$SOC : \frac{d^2AP(L)}{dL^2} = -2 < 0 \Rightarrow strictly\ concave \tag{5}$$

By checking the sign in second order, we find the interior solution in FOC gives us the maximum point. Also, the $AP(L)$ is a concave function in L.

From (4), we have:

$$\begin{cases} if\ L > 20, \ \rightarrow \frac{dAP(L)}{dL} < 0 \rightarrow AP(L)\ is\ decreasing\ in\ L \\ if\ L = 20, \ \rightarrow \frac{dAP(L)}{dL} = 0 \rightarrow AP(L)\ is\ horizontal\ in\ L \\ if\ L < 20, \ \rightarrow \frac{dAP(L)}{dL} > 0 \rightarrow AP(L)\ is\ increasing\ in\ L \end{cases} \tag{6}$$

Combine (3) & (6), we find that:

$$If\ MP(L) < AP(L) \Rightarrow AP(L)\ is\ decreasing\ inL \tag{7}$$

$$If\ MP(L) = AP(L) \Rightarrow AP(L)\ is\ horizontal\ inL \tag{8}$$

$$If\ MP(L) > AP(L) \Rightarrow AP(L)\ is\ increasing\ inL \tag{9}$$

This completes our proof.

2 Chapter 5.7

Question: A firm uses one input (L) to generate output (q) according to the production function $q = aL^b$, $a > 0$, and $b > 0$ (also $L \geq 0$). The input price is w and fixed costs are $c0$. Show that $\frac{dq}{dL}$ is rising if $\frac{dC}{dq}$ is falling, $\frac{dq}{dL}$ is falling if $\frac{dC}{dq}$ is rising, and $\frac{dq}{dL}$ neither rises nor falls if $\frac{dC}{dq}$ neither rises nor falls. How does your answer relate to the value of b? How does your result relate to the inverse function rule for differentiation?

Hints: Let's find the total cost function first:

$$C = fixed\ cost\ + variable\ cost\ = c0 + wL(q) \tag{10}$$

Now, we rewrite the production function as:

$$L = (\frac{q}{a})^{1/b} \tag{11}$$

Substitute (11) into (10), we have the total cost as a function of output:

$$C = c0 + w * (\frac{q}{a})^{1/b} \tag{12}$$

Therefore, we can characterize $\frac{dq}{dL}$ & $\frac{dC}{dq}$ by differentiating q w.r.t L & differentiating C w.r.t q respectively:

$$\frac{dq}{dL} = ab(L)^{b-1} \tag{13}$$

$$\frac{dC}{dq} = \frac{w}{ab}(\frac{q}{a})^{\frac{1-b}{b}} \tag{14}$$

Now, if $\frac{dC}{dq}$ is **falling** in q, this implies:

$$\frac{d^2C}{dq^2} = \frac{w(1-b)}{a^2b^2}(\frac{q}{a})^{\frac{1-2b}{b}} < 0 \tag{15}$$

Notice that all other terms besides $(1-b)$ are positive defined. In order to make $\frac{d^2C}{dq^2} < 0$, we need:

$$1 - b < 0 \Rightarrow b > 1 \tag{16}$$

Now, let's find the sign of $\frac{d^2q}{dL^2}$ conditional on $b > 1$:

$$\frac{d^2q}{dL^2}|_{b>1} = ab(b-1)L^{b-2}|_{b>1} > 0 \tag{17}$$

Where the greater sign follows $a > 0$, $b > 1$ & L is non-negative.

Combine (15), (16) & (17), we proved that $\frac{dq}{dL}$ is rising if $\frac{dC}{dq}$ is falling. Similar proof applies for proving $\frac{dq}{dL}$ is falling if $\frac{dC}{dq}$ is rising, and $\frac{dq}{dL}$ neither rises nor falls if $\frac{dC}{dq}$ neither rises nor falls.

In conclusion, If $b < 1$, then $\frac{dq}{dL}$ is decreasing in L and $\frac{dC}{dq}$ is increasing in q. If $b > 1$, then $\frac{dq}{dL}$ is increasing in L and $\frac{dC}{dq}$is decreasing in q. If $b = 1$ then $\frac{dq}{dL}$ and $\frac{dC}{dq}$ are neither increasing nor decreasing.