Lab 4 Solutions

February 3, 2017

1 Chapter 6.3 # f

Question: Find the stationary values of the following functions, and determine whether they give maxima, minima, or points of inflection.

$$y = 3x^4 - 10x^3 + 6x^2 + 5 \tag{1}$$

Hints: To get stationary points, we should derive first order condition.

$$Max : y = 3x^4 - 10x^3 + 6x^2 + 5 \tag{2}$$

$$FOC :\ 12x^3 - 30x^2 + 12x = 0 \tag{3}$$

$$\Rightarrow x(2x-1)(3x-6) = 0 \tag{4}$$

Therefore, we can find three stationary points, $x1 = 0$, $x2 = \frac{1}{2}$ & $x3 = 2$. Furthermore, we can determine the increasing and decreasing intervals of the objective function:

$$\begin{cases} x \in (-\infty,\ 0\ ],\ y' < 0,\ with\ x\ increase\, y\ decrease \\ x \in (0,\ \frac{1}{2}\ ],\ y' > 0,\ with\ x\ increase\, y\ increase \\ x \in (\frac{1}{2},\ 2\ ],\ y' < 0,\ with\ x\ increase\, y\ decrease \\ x \in (2,\ \infty\ ),\ y' > 0,\ with\ x\ increase\, y\ increase \end{cases} \tag{5}$$

The first-order condition $y'=f'(x^*) = 0$ only gives a necessary condition for x^* to yield an extreme value of the function. We should check second order to determine the convexity & concavity of the objective function to inference whether the solutions in first order condition gives us maximum, minimum or refection point.

$$SOC :\ y^{''} = 36x^2 - 60x + 12 \tag{6}$$

If we evaluate $y^{''}$ at 0, we have:

$$36x^2 - 60x + 12 = 0 \Rightarrow 3x^2 - 5x + 1 = 0 \tag{7}$$

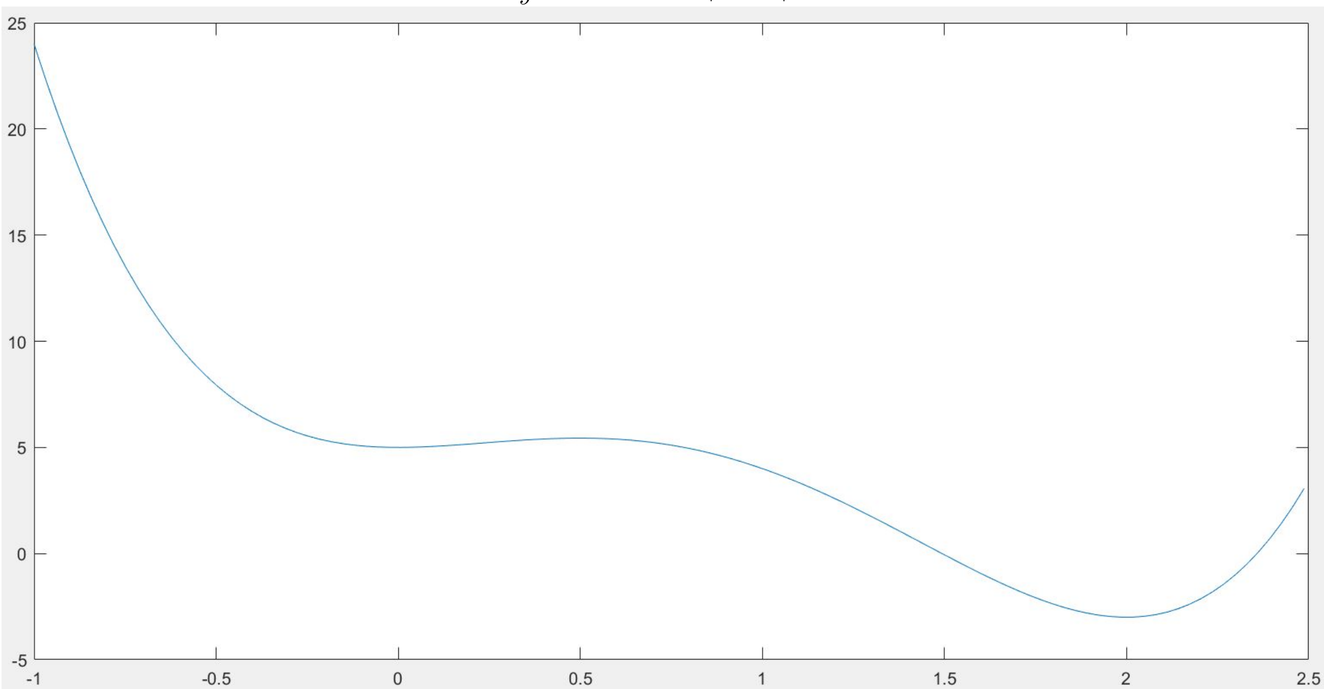
Solution of equation (7) is very simple and obtained by using the standard quadratic - root formula:

$$x_1 = \frac{5-\sqrt{13}}{6},\ x_2 = \frac{5+\sqrt{13}}{6} \tag{8}$$

Now, we can fully characterize the concave intervals and convex intervals of our objective function.

Figure 1

$y = 3x^4 - 10x^3 + 6x^2 + 5$



$$\begin{cases} x \in (-\infty,\ \frac{5-\sqrt{13}}{6}\ ],\ y^{''} > 0,\ y\ is\ convex\ in\ x \\ x \in (\frac{5-\sqrt{13}}{6}\ ,\ \frac{5+\sqrt{13}}{6}\ ],\ y^{''} < 0,\ y\ is\ concave\ in\ x \\ x \in (\frac{5+\sqrt{13}}{6}\ ,\ \infty,\ ],\ y^{''} > 0,\ y\ is\ convex\ in\ x \end{cases} \tag{9}$$

Since,

$$0 < \frac{5-\sqrt{13}}{6} < \frac{5-\sqrt{4}}{6} = \frac{1}{2} < \frac{5+\sqrt{13}}{6} < \frac{5+\sqrt{49}}{6} = 2 \tag{10}$$

Combine (5) & (9), we have:

$$\begin{cases} x \in (-\infty,\ 0\ ],\ y' < 0,\ convex \\ x \in (0, \frac{5-\sqrt{13}}{6}\ ],\ y' > 0,\ convex \\ x \in (\frac{5-\sqrt{13}}{6}\ ,\ \frac{1}{2}\ ],\ y' > 0,\ concave \\ x \in (\frac{1}{2}\ ,\ \frac{5+\sqrt{13}}{6}\ ],\ y' < 0,\ concave \\ x \in (\frac{5+\sqrt{13}}{6}\ ,\ 2],\ y' < 0,\ convex \\ x \in (2,\ \infty\ ),\ y' > 0,\ convex \end{cases} \tag{11}$$

Figure 1 shows the graph of this function.

Based on (11) & the figure 1, we can conclude that $x = 0$ is the local minimum, $x = \frac{1}{2}$ is the local maximum, $x = 2$ is the global minimum (Since $y|_{x=2} < y|_{x=1/2}$).

2 Chapter 6.3 # f Continue

Question: Now, we add one more restriction into our function, that is $x \in [\frac{2}{5}, 1.5]$. Solve this constrained minimization problem & derive the minimum point. If the solution is an endpoint, find the shadow price.

Hints: With constraint, our problem becomes:

$$Max: \; y = 3x^4 - 10x^3 + 6x^2 + 5 \;\; subject \; to \; x \in [\frac{2}{5}, 1.5] \tag{12}$$

Contingent on our discussion in previews question, we can determine the minimum point could only be one of the two endpoints of our constraint (why? try to draw a graph including a restriction).

Substitute $x = \frac{2}{5}$ into y:

$$y = f(x)|_{x=\frac{2}{5}} = 5.3968 \tag{13}$$

Substitute $x = 1.5$ into y:

$$y = f(x)|_{x=1.5} = -0.0625 \tag{14}$$

Compare (12) & (13), we find that at $x = 1.5$, y achieves the minimum.

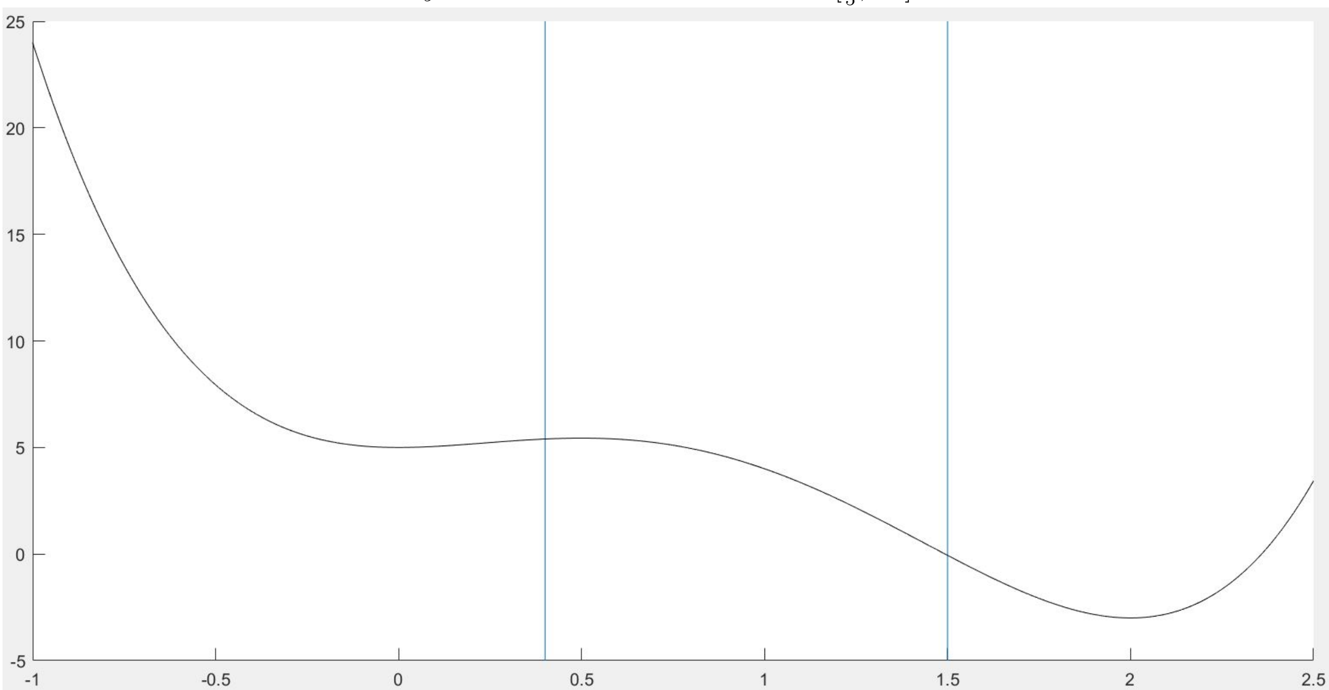
Now, let's calculate the shadow price of this restriction:

$$y^{'}|_{x=1.5} = (12x^3 - 30x^2 + 12x)|_{x=1.5} = -9 \tag{15}$$

Intuitively, the shadow price measures marginal value of a relaxation in our restriction. Figure 2 shows the graph with our new constraint. The restriction area is bounded by two vertical blue line. It is clear that the restricted function achieves the minimum at $x = 1.5$.

Figure 2

$y = 3x^4 - 10x^3 + 6x^2 + 5$ with $x \in [\frac{2}{5}, 1.5]$



At home, you may try finding the minimum with $x \in [-0.7, 0.5]$. In this case, will the minimized point be the endpoint or stationary point?

3 Chapter 6.9

Question: A student is preparing for exams in two subjects. She estimates that the grades she will obtain in each subject, as a function of the amount of time spent working on them are:

$$g_1 = 20 + 20\sqrt{t_1} \tag{16}$$

$$g_2 = -80 + 3t_2 \tag{17}$$

where g_i is the grade in subject i and t_i is the number of hours per week spent in studying for subject i, $i = 1, 2$.

She wishes to maximize her grade average $(g_1 + g_2)/2$. She cannot spend in total more than 60 hours studying in the week. Find the optimal values of t_1 and t_2 and discuss the characteristics of the solution. Why is this essentially an economic problem? [Hint: Assume that 60 hours a week is a binding constraint and express the problem as one involving t_1 only.]

Hints: At optimal, this student must **use out** all available hours, that is the time constraint is binding. Otherwise she can increase grade simply by increasing more study time.

$$t_1 + t_2 = 60 \tag{18}$$

Rewrite (18), we have:

$$t_2 = 60 - t_1 \tag{19}$$

Now, substitute (19) into (17), we find g_2 becomes a function about t_1 only:

$$g_2 = -80 + 3(60 - t_1) = 100 - 3t_1 \tag{20}$$

Therefore, our optimization problem becomes:

$$Max: \ \pi = (g_1 + g_2)/2 \tag{21}$$

Substitute (17), (20) into (21), the problem becomes maximizing objective function by choosing t_1:

$$Max: \ \pi = (20 + 20\sqrt{t_1} + 100 - 3t_1)/2 = 10\sqrt{t_1} - \frac{3}{2}t_1 + 60 \tag{22}$$

$$subject \ to \ t_1 \in [0 \ , \ 60 \ ] \tag{23}$$

Now, let's derive the first and second order condition to characterize the graph of our objective function.

$$FOC: \frac{d\pi}{dt_1} = 5t_1^{-1/2} - \frac{3}{2} = 0 \Rightarrow t_1 = 100/9 \tag{24}$$

$$\Rightarrow \begin{cases} t_1 \in [0 \ , \ 100/9 \ ], \ \frac{d\pi}{dt_1} > 0, \ \pi \ is \ increasing \ in \ t_1 \\ t_1 \in (100/9 \ , \ 60 \ ], \ \frac{d\pi}{dt_1} > 0, \ \pi \ is \ decreasing \ in \ t_1 \end{cases} \tag{25}$$

$$SOC: \ \frac{d^2\pi}{dt_1^2} = -5/2 < 0 \tag{26}$$

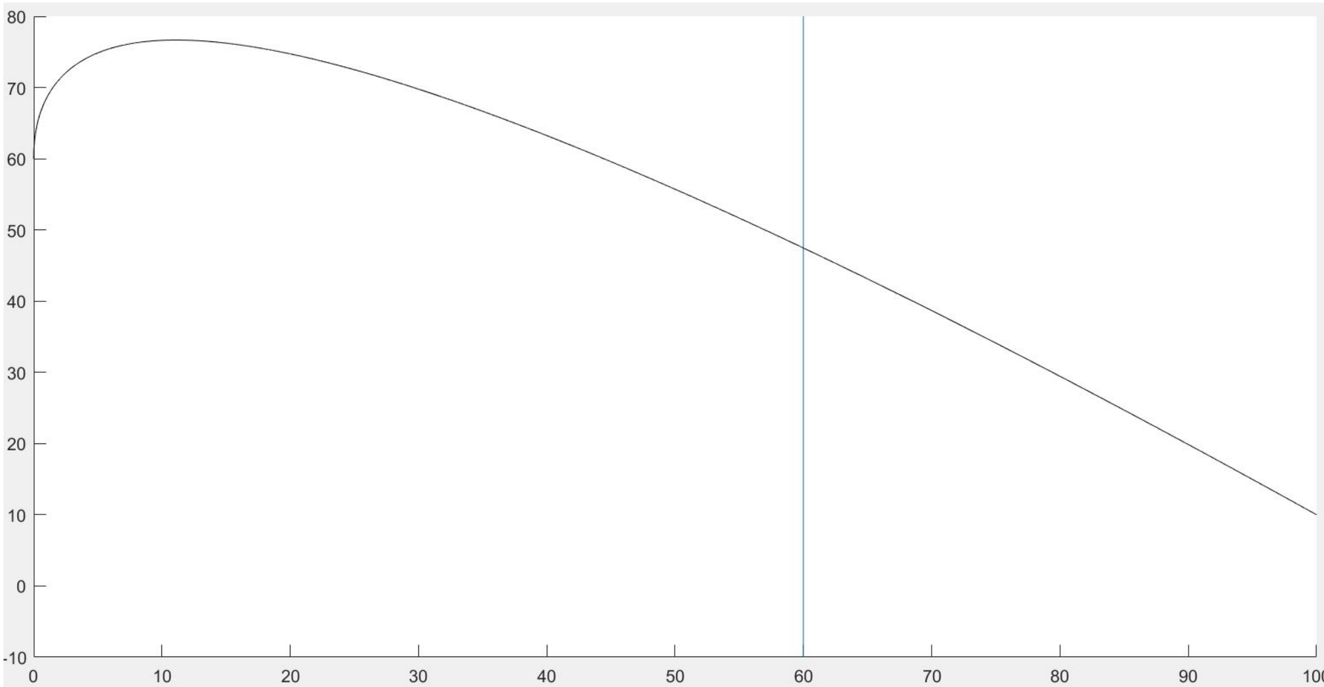
Equation (26) shows the function is concave in t_1 for all $t_1 \in [0 \ , \ 60 \ ]$. Therefore, the solution we derived in FOC (24) is a global maximization point. Figure 3 gives the graph of this function.

In summary

$$\begin{cases} t_1^* = 100/9 \\ t_2^* = 60 - 100/9 = 440/9 \end{cases} \tag{27}$$

Figure 3

$\pi = 10\sqrt{t_1} - \frac{3}{2}t_1 + 60$ with $x \in [0\ ,\ 60\ ]$



Intuitively, this is a resource allocation problem. Since the resource (time) is scarce, this question investigates how to allocate it between two ways to maximize agent's the objective function. Economically, the agent will maximize its 'utility' (that is π in our question) by equaling the marginal 'utility' of the first subject with the marginal 'utility' of the second subject. That is:

$$MU_1 = \frac{\partial(\frac{(g_1+g_2)}{2})}{\partial t_1} = MU_2 = \frac{\partial(\frac{(g_1+g_2)}{2})}{\partial t_2} \tag{28}$$

$$\Rightarrow \frac{\partial(\frac{20+20\sqrt{t_1}-80+3t_2}{2})}{\partial t_1} = 5t_1^{-1/2} = \frac{\partial(\frac{20+20\sqrt{t_1}-80+3t_2}{2})}{\partial t_2} = 3/2 \tag{29}$$

$$\Rightarrow 5t_1^{-1/2} = 3/2 \tag{30}$$

Compare (24) & (30), we find that we derived the same solution but only depends on the economics intuition.